$$\boxed{\operatorname{rot}\mathbf{B} = \mu_0 \mathbf{j}}$$

$$\int\limits_{S_0} \operatorname{rot}\mathbf{B}\, d\mathbf{S} = \mu_0 \int\limits_{S_0} \mathbf{j}\, d\mathbf{S}$$

$$\boxed{\oint\limits_{L} \mathbf{B}\, d\mathbf{l} = \mu_0 \sum_{k=1}^{N} I_k}$$

Полная система уравнений для индукции магнитного поля

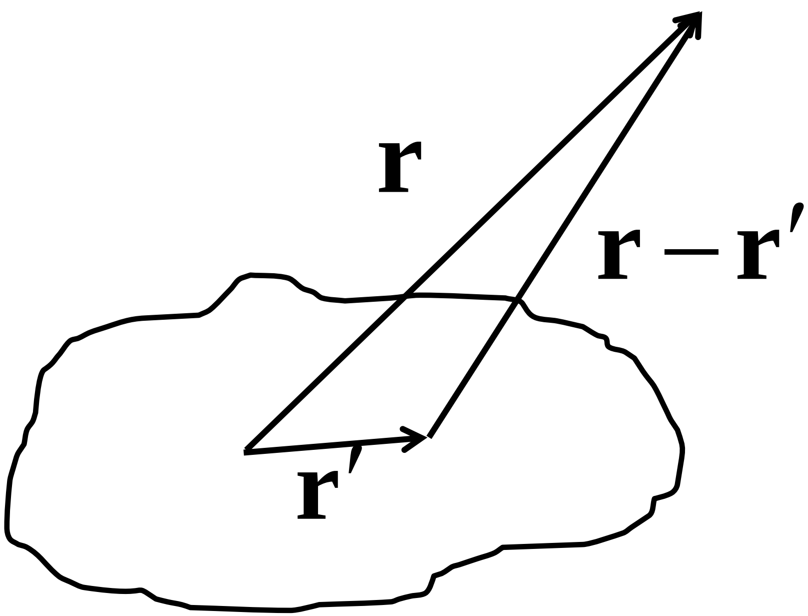
$$\boxed{\operatorname{rot}\mathbf{B} = \mu_0 \mathbf{j} \qquad \operatorname{div}\mathbf{B} = 0}$$

Граничные условия

$$\mathbf{B}_{2t} - \mathbf{B}_{1t} = \mu_0 \mathbf{i} \qquad \mathbf{B}_{2n} = \mathbf{B}_{1n}$$

Векторный потенциал элементарного тока на больших расстояниях

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int\limits_V \frac{\mathbf{j}(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$



$$\frac{1}{r-r'} \approx \frac{1}{r} + \frac{(\mathbf{r}'\cdot\mathbf{r})}{r^3} + \ldots$$

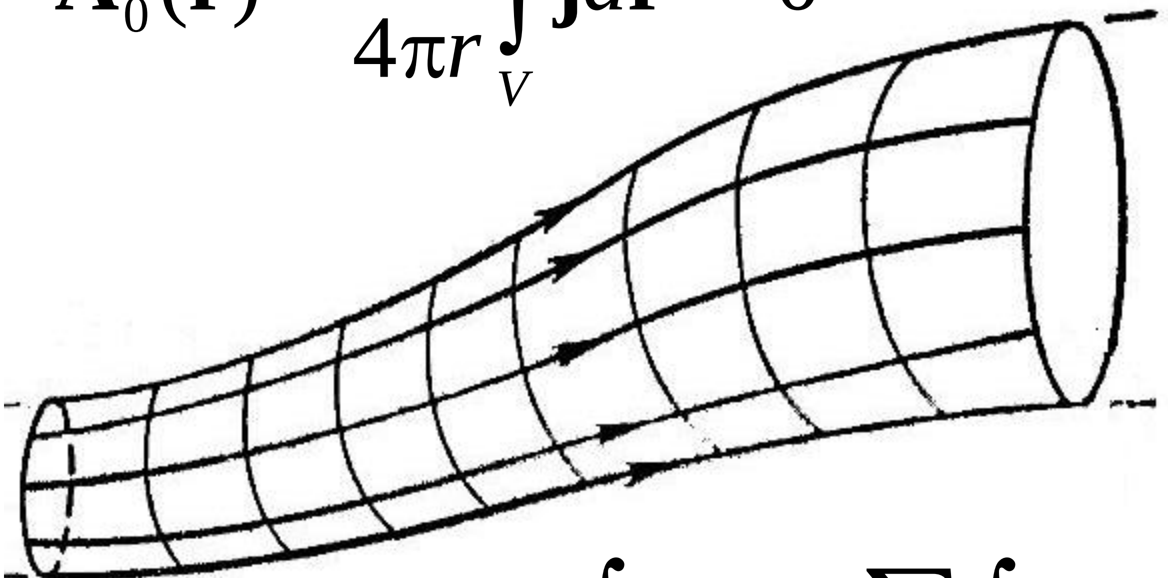
$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_0(\mathbf{r}) + \mathbf{A}_1(\mathbf{r}) + \ldots =$$

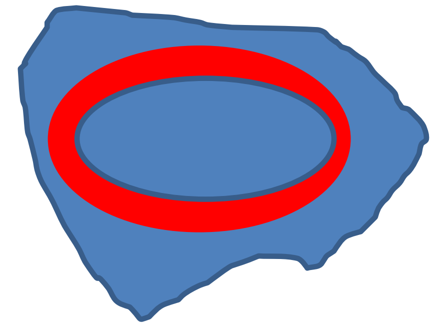
$$= \frac{\mu_0}{4\pi r}\int\limits_V \mathbf{j}d\mathbf{r}' +$$

$$+ \frac{\mu_0}{4\pi r^3}\int\limits_V \mathbf{j}(\mathbf{r}'\cdot\mathbf{r})d\mathbf{r}' + \ldots$$

$$\mathbf{A}_0(\mathbf{r}) = \frac{\mu_0}{4\pi r}\int_V \mathbf{j}\,d\mathbf{r}' = 0$$

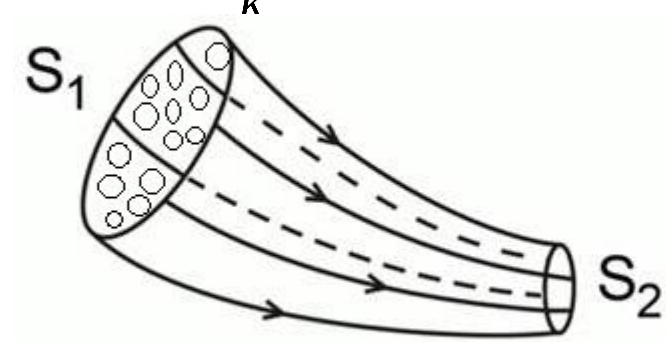
$$\operatorname{div}\mathbf{j} = 0$$





$$\int_V \mathbf{j}\,d\mathbf{r}' = \sum_k \int_{L_k} I_k\,d\mathbf{l}_k = \sum_k I_k \oint d\mathbf{l}_k = 0$$

$$\mathbf{A} = \mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{4\pi r^3}\int_V \mathbf{j}(\mathbf{r}'\cdot\mathbf{r})\,d\mathbf{r}'$$



$$\mathbf{j}(\mathbf{r}'\cdot\mathbf{r}) = \frac{1}{2}\{\mathbf{j}(\mathbf{r}'\cdot\mathbf{r}) - \mathbf{r}'(\mathbf{j}\cdot\mathbf{r})\} + \frac{1}{2}\{\mathbf{j}(\mathbf{r}'\cdot\mathbf{r}) + \mathbf{r}'(\mathbf{j}\cdot\mathbf{r})\}$$

$$\mathbf{j}(\mathbf{r}'\cdot\mathbf{r})-\mathbf{r}'(\mathbf{j}\cdot\mathbf{r})=-[\mathbf{r}\times[\mathbf{r}'\times\mathbf{j}]]=[[\mathbf{r}'\times\mathbf{j}]\times\mathbf{r}]$$

$$\mathbf{A}_1(\mathbf{r})=\frac{\mu_0}{8\pi r^3}\int_V[[\mathbf{r}'\times\mathbf{j}(\mathbf{r}')]\times\mathbf{r}]d\mathbf{r}'+$$

$$+\frac{\mu_0}{8\pi r^3}\int_V\{\mathbf{j}(\mathbf{r}'\cdot\mathbf{r})+\mathbf{r}'(\mathbf{j}\cdot\mathbf{r})\}d\mathbf{r}'$$

Покажем, что второй интеграл равен нулю

$$\mathbf{a}\cdot\int_V\{\mathbf{j}(\mathbf{r}'\cdot\mathbf{r})+\mathbf{r}'(\mathbf{j}\cdot\mathbf{r})\}d\mathbf{r}'=$$

$$=\int_V\{(\mathbf{a}\cdot\mathbf{j})(\mathbf{r}'\cdot\mathbf{r})+(\mathbf{a}\cdot\mathbf{r}')(\mathbf{j}\cdot\mathbf{r})\}d\mathbf{r}'=$$

$$=\int_V\{\mathbf{j}\,\mathrm{grad}'(\mathbf{a}\cdot\mathbf{r}')(\mathbf{r}'\cdot\mathbf{r})+(\mathbf{a}\cdot\mathbf{r}')\mathbf{j}\cdot\mathrm{grad}'(\mathbf{r}'\cdot\mathbf{r})\}d\mathbf{r}'=$$

$$=\int_V\mathbf{j}\cdot\mathrm{grad}'\{(\mathbf{a}\cdot\mathbf{r}')(\mathbf{r}'\cdot\mathbf{r})\}d\mathbf{r}'\quad\Rightarrow$$

$$\operatorname{div}'(\gamma \times \mathbf{b}) = \gamma \operatorname{div}' \mathbf{b} + \mathbf{b} \operatorname{grad}' \gamma$$

$$\int_V \mathbf{j} \cdot \operatorname{grad}'\{(\mathbf{a} \cdot \mathbf{r}')(\mathbf{r}' \cdot \mathbf{r})\} d\mathbf{r}' = \int_V \operatorname{div}'\{(\mathbf{a} \cdot \mathbf{r}')(\mathbf{r}' \cdot \mathbf{r})\mathbf{j}\} d\mathbf{r}' -$$

$$-\int_V \underbrace{(\mathbf{a} \cdot \mathbf{r}')(\mathbf{r}' \cdot \mathbf{r}) \operatorname{div}' \mathbf{j} d\mathbf{r}'}_{=0} = \underbrace{\int_S j_n (\mathbf{a} \cdot \mathbf{r}')(\mathbf{r}' \cdot \mathbf{r}) dS}_{=0} = 0$$

$$(\operatorname{div}' \mathbf{j} = 0) \qquad (j_n = 0)$$

$\mathbf{a}$ - произвольный вектор

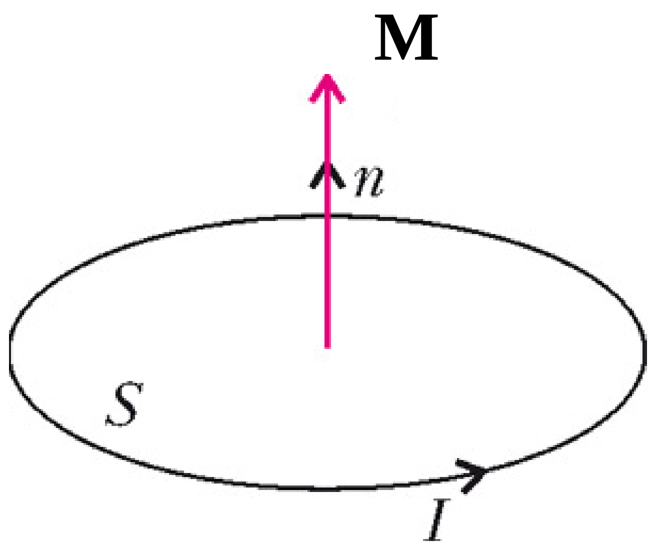
$$\int_V \{\mathbf{j}(\mathbf{r}' \cdot \mathbf{r}) + \mathbf{r}'(\mathbf{j} \cdot \mathbf{r})\} d\mathbf{r}' = 0$$

$$\mathbf{A}_1(\mathbf{r}) = \frac{\mu_0}{8\pi r^3} \int_V [[\mathbf{r}' \times \mathbf{j}(\mathbf{r}')] \times \mathbf{r}] d\mathbf{r}' = \frac{\mu_0}{4\pi r^3} \int_V [\mathbf{J}(\mathbf{r}') \times \mathbf{r}] d\mathbf{r}' =$$

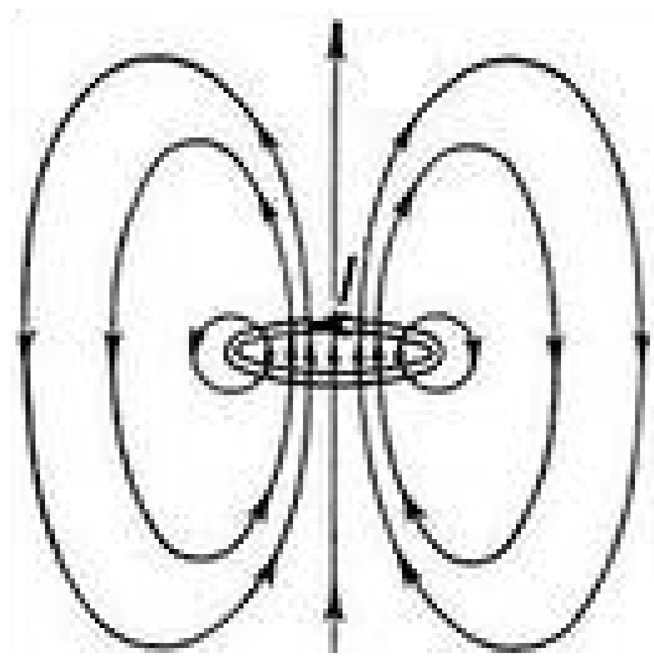
$$= \frac{\mu_0}{4\pi} \frac{[\mathbf{M} \times \mathbf{r}]}{r^3}$$

$$\mathbf{J}(\mathbf{r}') = \frac{1}{2}[\mathbf{r}' \times \mathbf{j}(\mathbf{r}')] \qquad \mathbf{M} = \int_V \mathbf{J}(\mathbf{r}') d\mathbf{r}' = \frac{1}{2} \int_V [\mathbf{r}' \times \mathbf{j}] d\mathbf{r}'$$

$$\mathbf{M} = \int_V \mathbf{J}(\mathbf{r}')d\mathbf{r}' = \frac{1}{2}\int_V [\mathbf{r}' \times \mathbf{j}]d\mathbf{r}' = \frac{I}{2}\int_L [\mathbf{r}' \times d\mathbf{l}]$$



$$\mathbf{M} = IS\mathbf{n}$$



Индукция магнитного поля на больших расстояниях равна

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left\{\frac{3\mathbf{r}(\mathbf{Mr})}{r^5} - \frac{\mathbf{M}}{r^3}\right\}$$

Магнитное поле совокупности элементарных токов на больших расстояниях

$$\mathbf{B} = \operatorname{rot}\mathbf{A} = \frac{\mu_0}{4\pi}\operatorname{rot}\frac{[\mathbf{M}\times\mathbf{r}]}{r^3} =$$

$$= \frac{\mu_0}{4\pi}\left\{\frac{1}{r^3}\operatorname{rot}[\mathbf{M}\times\mathbf{r}] + \left[\operatorname{grad}\left(\frac{1}{r^3}\right)\times[\mathbf{M}\times\mathbf{r}]\right]\right\}$$

$$\operatorname{rot}[\mathbf{M}\times\mathbf{r}] = \underbrace{(\mathbf{r}\operatorname{grad})\mathbf{M}}_{=0} - \underbrace{(\mathbf{M}\operatorname{grad})\mathbf{r}}_{=\mathbf{M}} + \underbrace{\mathbf{M}\operatorname{div}\mathbf{r}}_{=3\mathbf{M}} - \underbrace{\mathbf{r}\operatorname{div}\mathbf{M}}_{=0} = 2\mathbf{M}$$

$$\left[\operatorname{grad}\left(\frac{1}{r^3}\right)\times[\mathbf{M}\times\mathbf{r}]\right] = -\frac{3}{r^5}[\mathbf{r}\times[\mathbf{M}\times\mathbf{r}]] = -\frac{3\mathbf{M}}{r^3} + \frac{3\mathbf{r}(\mathbf{M}\cdot\mathbf{r})}{r^5}$$

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left\{\frac{3\mathbf{r}(\mathbf{M}\mathbf{r})}{r^5} - \frac{\mathbf{M}}{r^3}\right\}$$

Магнитное поле в среде

$$\operatorname{rot}\mathbf{B}_m = \mu_0 \mathbf{j}_m = \mu_0 \rho \mathbf{v}_m$$

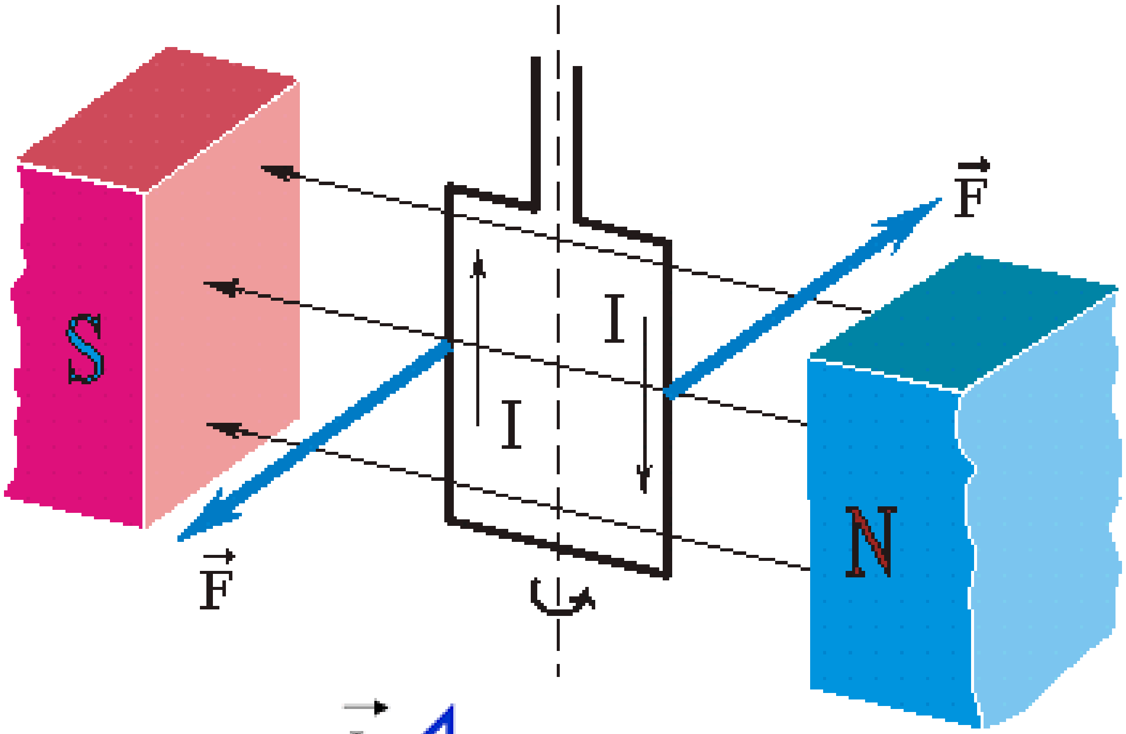
$$\operatorname{div}\rho\mathbf{v}_m = 0 \qquad \operatorname{div}\mathbf{B}_m = 0$$

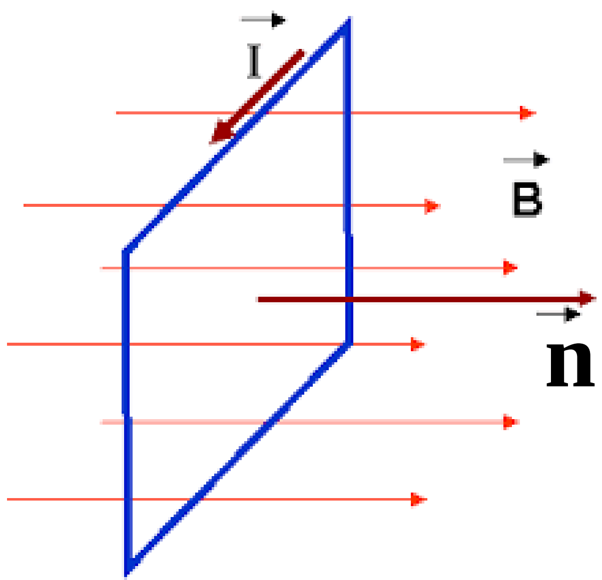
$$\overline{f} = \frac{1}{2\tau}\int_{-\tau}^{\tau}\frac{1}{v_0}\int_{v_0} f_m\,dv\,dt$$

$$\operatorname{rot}\mathbf{B} = \mu_0\overline{\rho\mathbf{v}_m} \qquad \operatorname{div}\mathbf{B} = 0 \qquad \operatorname{div}\overline{\rho\mathbf{v}_m} = 0$$

$$\overline{\rho\mathbf{v}_m} = \mathbf{j} + \mathbf{j}_{\mathrm{m}}$$

Элементарный ток в магнитном поле

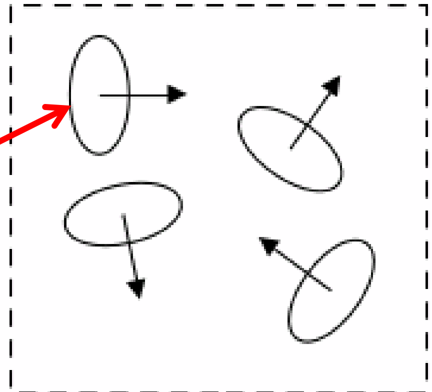


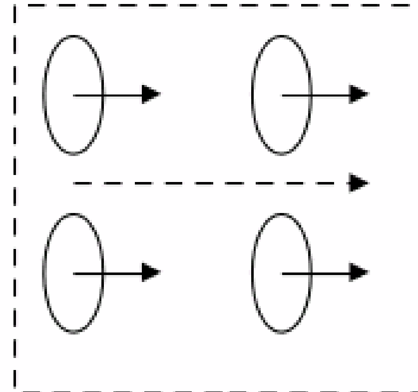


$$\mathbf{B} \uparrow\uparrow \mathbf{M} = IS\mathbf{n}$$

Молекулярный ток

$\mathbf{j}_m$





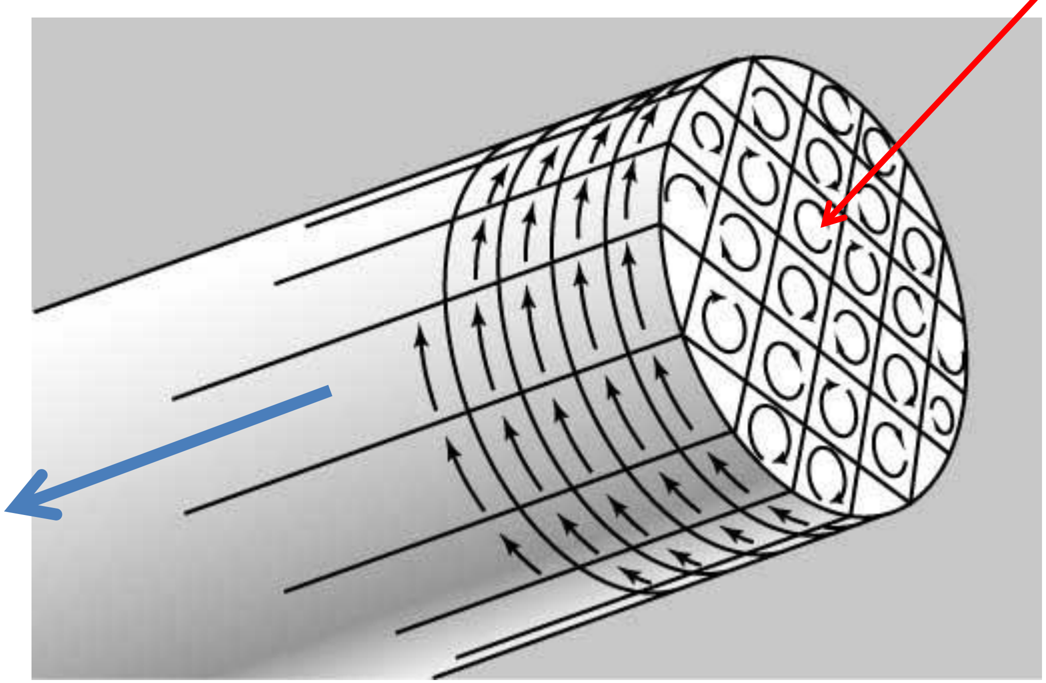
$\mathbf{j}_m$

$$\mathbf{M} = \int_V \mathbf{J}(\mathbf{r}')d\mathbf{r}' =$$

$$= \frac{1}{2}\int_V [\mathbf{r}' \times \mathbf{j}]d\mathbf{r}'$$

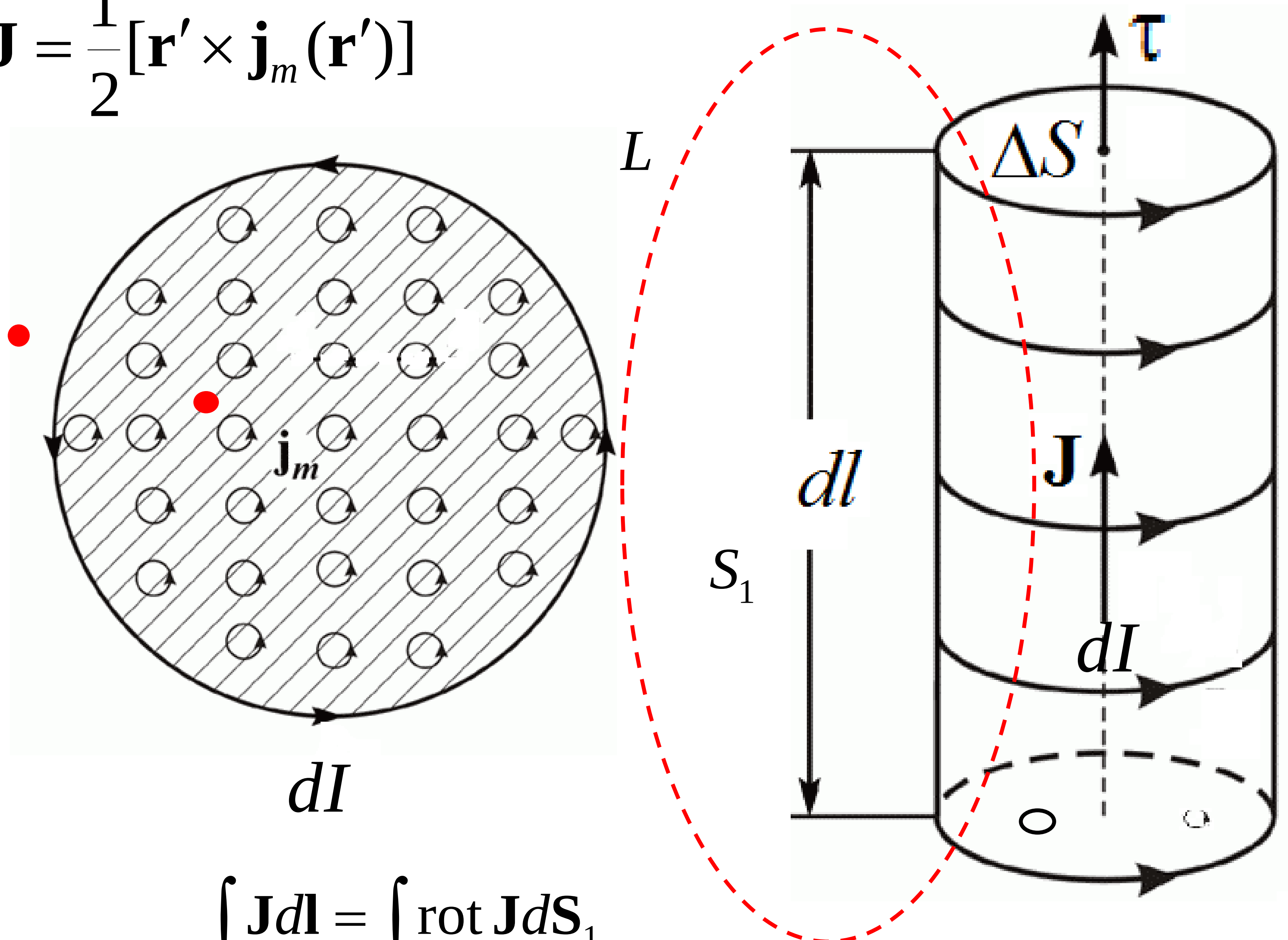
$$\mathbf{M} = \int \mathbf{J}dv$$

J –плотность дипольного магнитного момента

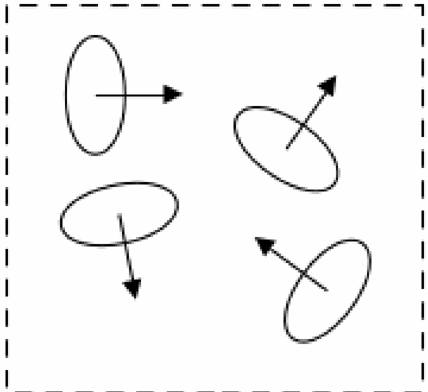


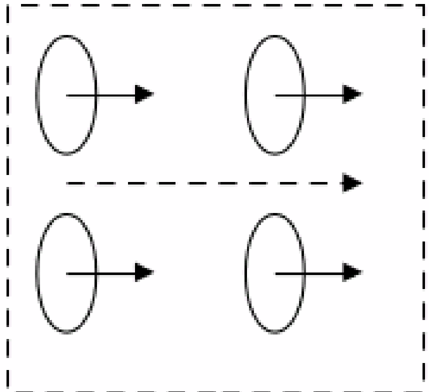
$$\mathbf{J} = \alpha\mathbf{B}$$

$$\mathbf{J} = \frac{1}{2}[\mathbf{r}' \times \mathbf{j}_m(\mathbf{r}')]$$



$$\int_L \mathbf{J}d\mathbf{l} = \int_{S_1} \mathrm{rot}\,\mathbf{J}d\mathbf{S}_1$$

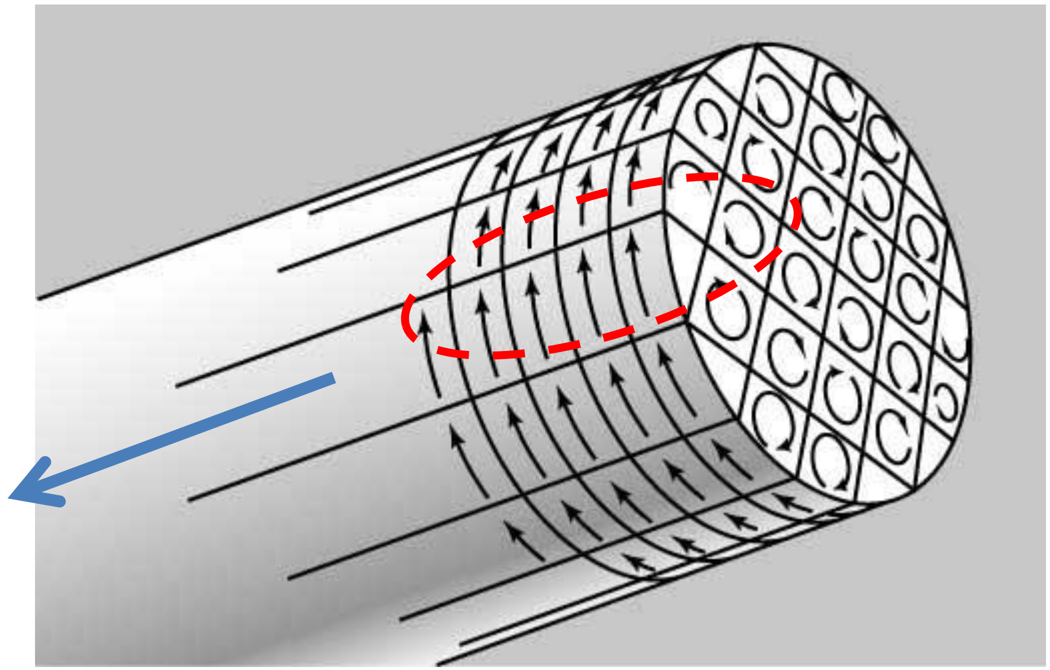




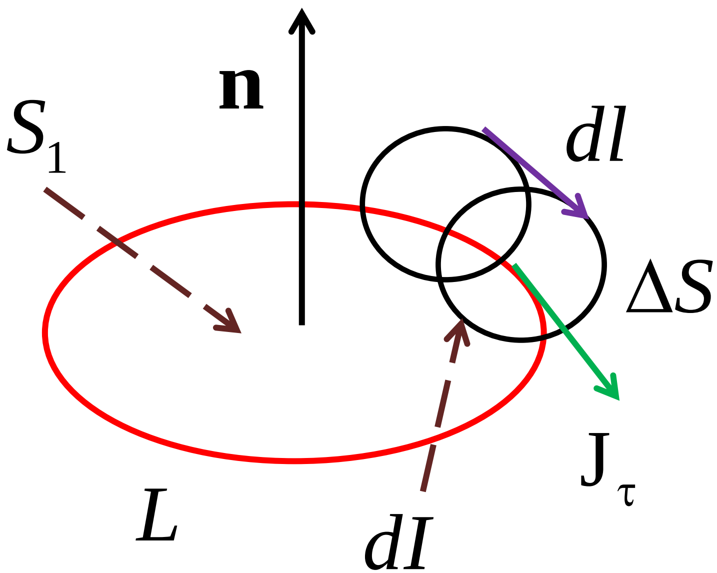
$$\mathbf{M} = \int \mathbf{J} dv$$

Преобразуем соотношение

$$\int_L \mathbf{J} d\mathbf{l} = \int_{S_1} \operatorname{rot} \mathbf{J} d\mathbf{S}_1$$

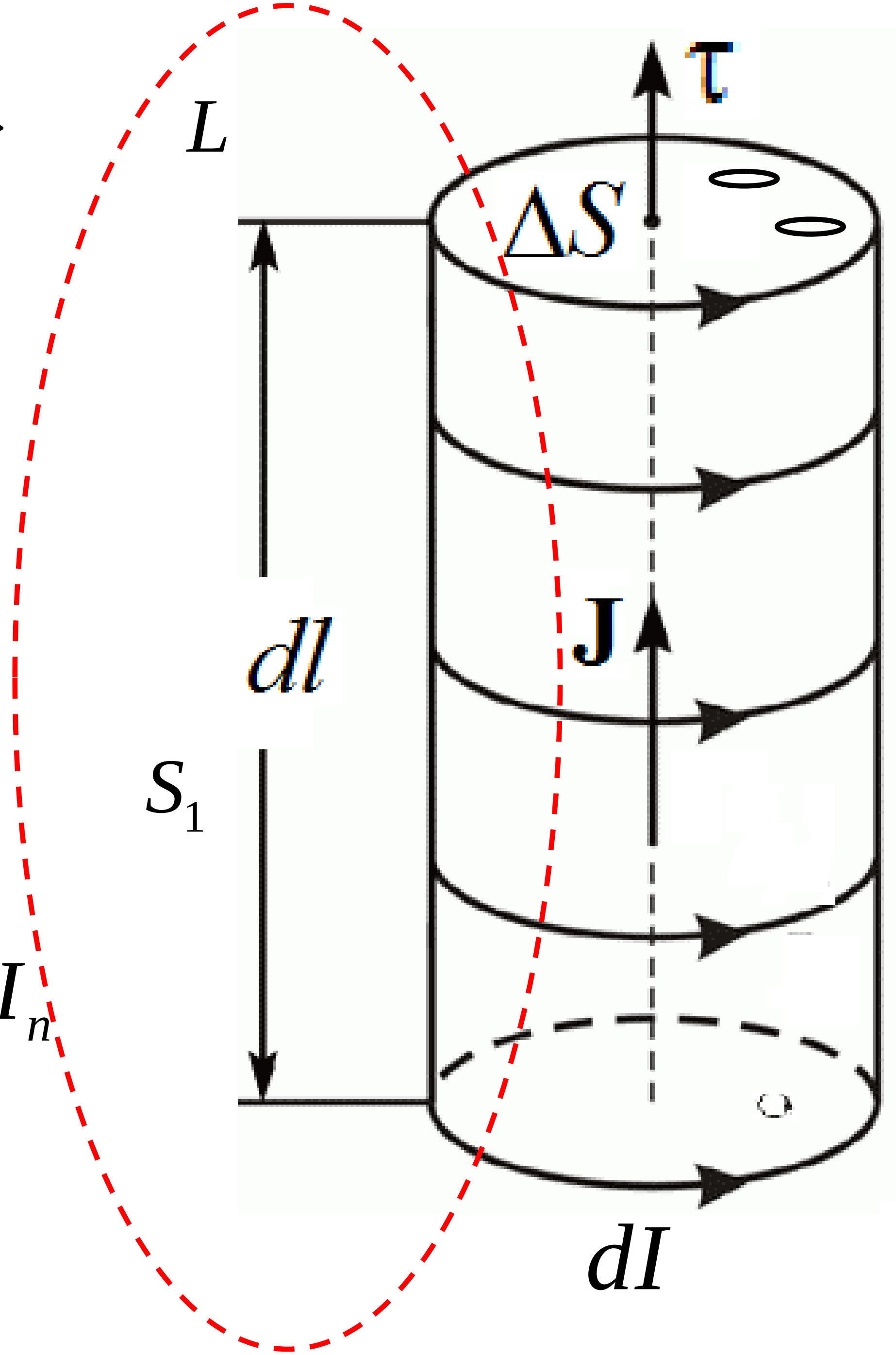


$$\int_L \mathbf{J}d\mathbf{l} = \int_L \mathrm{J}_\tau \, dl \frac{\Delta S}{\Delta S} = \int_L \mathrm{J}_\tau \frac{dV}{\Delta S} \Rightarrow$$



$$\Rightarrow \int_L \frac{dM_\tau}{\Delta S} = \int_L \frac{dI_n \Delta S}{\Delta S} = \int_L dI_n$$

$$= I = \int_{S_1} \mathbf{j}_m d\mathbf{S}_1 = \int_{S_1} \mathrm{rot}\,\mathbf{J} d\mathbf{S}_1$$



$$\boxed{\mathbf{j}_m = \operatorname{rot}\mathbf{J}}$$

$$\operatorname{rot}\mathbf{B} = \mu_0(\mathbf{j} + \mathbf{j}_m) = \mu_0\mathbf{j} + \mu_0 \operatorname{rot}\mathbf{J}$$

$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{J} = \mathbf{B}/\mu_0 - \alpha\mathbf{B} \leftarrow \mathbf{J} = \alpha\mathbf{B} \qquad \mu = (1 - \alpha\mu_0)^{-1}$$

$$\operatorname{rot}\mathbf{H} = \mathbf{j} \qquad \operatorname{div}\mathbf{B} = 0 \qquad \mathbf{B} = \mu\mu_0\mathbf{H}$$

Граничные условия

$$\mathbf{H}_{2t} - \mathbf{H}_{1t} = \mathbf{i} \qquad \mathbf{B}_{2n} - \mathbf{B}_{1n} = 0$$

Единственность решения системы уравнений для индукции и напряженности магнитного поля

$$\mathbf{B} - \mathbf{B}' = \mathbf{B}'' \qquad \mathbf{H} - \mathbf{H}' = \mathbf{H}''$$

Единственность решения системы уравнений для индукции магнитного поля

$$\operatorname{rot}\mathbf{H}'' = 0 \qquad \operatorname{div}\mathbf{B}'' = 0 \qquad \mathbf{H}''_{2t} = \mathbf{H}''_{1t} \qquad \mathbf{B}''_{2n} = \mathbf{B}''_{1n}$$

$$\operatorname{div}[\mathbf{a}\times\mathbf{b}] = \mathbf{b}\operatorname{rot}\mathbf{a} - \mathbf{a}\operatorname{rot}\mathbf{b}$$

$$\int \mathbf{b}\operatorname{rot}\mathbf{a}\,dv = \int \operatorname{div}[\mathbf{a}\times\mathbf{b}]\,dv + \int \mathbf{a}\operatorname{rot}\mathbf{b}\,dv$$

$$\mathbf{b} = \mathbf{H}'' \qquad \mathbf{a} = \mathbf{A}'' \qquad \mathbf{B}'' = \operatorname{rot}\mathbf{A}''$$

$$\int_V \mathbf{H}''\cdot\mathbf{B}''dv = \frac{1}{\mu\mu_0}\int_V \operatorname{div}[\mathbf{A}''\times\mathbf{B}'']dv + \underbrace{\int_V \mathbf{A}''\operatorname{rot}\mathbf{H}''dv}_{=0} =$$

$$= \frac{1}{\mu\mu_0}\int_S \underbrace{[\mathbf{A}''\times\mathbf{B}'']_n}_{\sim 1/r^3}\,dS \Rightarrow 0$$

$$\Delta\mathbf{A} = -\mu_0(\mathbf{j} + \mathbf{j}_m) = -\mu_0\mathbf{j} - \mu_0 \operatorname{rot}\mathbf{J} =$$

$$= -\mu_0\mathbf{j} - \alpha\mu_0 \operatorname{rot}\mathbf{B} = -\mu_0\mathbf{j} - \alpha\mu_0 \operatorname{rotrot}\mathbf{A} =$$

$$= -\mu_0\mathbf{j} + \alpha\mu_0\Delta\mathbf{A}$$

$$\operatorname{div}\mathbf{A} = 0$$

$$\mu = (1 - \alpha\mu_0)^{-1}$$

Уравнение Пуассона

$$\Delta\mathbf{A} = -\mu_0\mu\mathbf{j} \quad \Rightarrow \quad \mathbf{A}(\mathbf{r}) = \frac{\mu_0\mu}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}$$

$$\mathbf{B}(\mathbf{r}) = \operatorname{rot}\mathbf{A}(\mathbf{r}) = \frac{\mu_0\mu}{4\pi}\int_V \operatorname{rot}\left\{\frac{\mathbf{j}(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}\right\} =$$

$$= \frac{\mu_0\mu}{4\pi}\int_V \frac{[\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')]d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\operatorname{rot}\varphi\mathbf{a} = \varphi\operatorname{rot}\mathbf{a} + [\operatorname{grad}\varphi\times\mathbf{a}]$$

Сила действующая на элементарный ток

$$\mathbf{F} = \int\limits_V [\mathbf{j} \times \mathbf{B}] dv$$

$$\mathbf{F} = \int\limits_V [\mathbf{j} \times \mathbf{B}_0] dv = -\left[\mathbf{B}_0 \times \cancelto{=0}{\int\limits_V \mathbf{j} dv} \right] = 0$$

$$B_i = B_i(0) + x\left(\frac{\partial B_i}{\partial x}\right)_0 + y\left(\frac{\partial B_i}{\partial y}\right)_0 + z\left(\frac{\partial B_i}{\partial z}\right)_0$$

$$F_z = \int\limits_V j_x \left(x\left(\frac{\partial B_y}{\partial x}\right)_0 + y\left(\frac{\partial B_y}{\partial y}\right)_0 + z\left(\frac{\partial B_y}{\partial z}\right)_0 \right) dv -$$

$$- \int\limits_V j_y \left(x\left(\frac{\partial B_x}{\partial x}\right)_0 + y\left(\frac{\partial B_x}{\partial y}\right)_0 + z\left(\frac{\partial B_x}{\partial z}\right)_0 \right) dv =$$

Сила действующая на элементарный ток

$$F_z = \left(\frac{\partial B_y}{\partial x}\right)_0 \cdot \int_V \underline{j_x x dv} + \left(\frac{\partial B_y}{\partial y}\right)_0 \cdot \int_V j_x y dv +$$

$$+\left(\frac{\partial B_y}{\partial z}\right)_0 \cdot \int_V j_x z dv - \left(\frac{\partial B_x}{\partial x}\right)_0 \cdot \int_V j_y x dv -$$

$$-\left(\frac{\partial B_x}{\partial y}\right)_0 \cdot \int_V \underline{j_y y dv} - \left(\frac{\partial B_x}{\partial z}\right)_0 \cdot \int_V j_y z dv$$

$$\int_V j_y y dv = \int_V j_x x dv = 0$$

$$\varphi = x^2 / 2$$

$$\operatorname{div}(\varphi \mathbf{a}) = \varphi \operatorname{div} \mathbf{a} + \mathbf{a} \operatorname{grad} \varphi$$

$$\mathbf{a} = \mathbf{j}$$

$$x j_x = \frac{1}{2}\operatorname{div}(x^2 \mathbf{j}) - \cancel{\frac{x^2}{2}\operatorname{div}\mathbf{j}} = \frac{1}{2}\operatorname{div}(x^2 \mathbf{j})$$

$$\int\limits_V j_x x\,dv = \frac{1}{2}\int\limits_V \operatorname{div}(x^2\mathbf{j})\,dv = \frac{1}{2}\oint\limits_{S_{провод.}} x^2 j_n\,dS = 0$$

$$\int\limits_V j_y y\,dv = 0 \qquad\qquad \mathbf{M} = \frac{1}{2}\int\limits_V [\mathbf{r}\times\mathbf{j}]\,d\mathbf{r}$$

$$\int\limits_V x j_y\,dv = \underbrace{\frac{1}{2}\int\limits_V (x j_y - y j_x)\,dv}_{= M_z} + \underbrace{\frac{1}{2}\int\limits_V (x j_y + y j_x)\,dv}_{= 0} = M_z$$

$$x j_y + y j_x = \mathbf{j}\operatorname{grad}(xy) \qquad \operatorname{div}(\varphi\mathbf{a}) = \varphi\operatorname{div}\mathbf{a} + \mathbf{a}\operatorname{grad}\varphi$$

$$\int\limits_V (x j_y + y j_x)\,dv = \int\limits_V \mathbf{j}\operatorname{grad}(xy)\,dv =$$

$$= \int\limits_V \operatorname{div}(xy\mathbf{j})\,dv - \underbrace{\int\limits_V xy\operatorname{div}\mathbf{j}\,dv}_{= 0\ (\operatorname{div}\mathbf{j} = 0)} = \int\limits_{S_{провод.}} xy j_n\,dS = 0$$

Аналогично $$\int\limits_V y j_x dv = -M_z$$

$$\int\limits_V y j_z dv = -\int\limits_V z j_y dv = M_x \qquad \int\limits_V z j_x dv = -\int\limits_V x j_z dv = M_y$$

$$F_z = -M_z\left(\frac{\partial B_y}{\partial y}\right) + M_y\left(\frac{\partial B_y}{\partial z}\right) - M_z\left(\frac{\partial B_x}{\partial x}\right) + M_x\left(\frac{\partial B_x}{\partial z}\right) =$$

$$= M_x\left(\frac{\partial B_x}{\partial z}\right) + M_y\left(\frac{\partial B_y}{\partial z}\right) + M_z\left(\frac{\partial B_z}{\partial z}\right) = \frac{\partial}{\partial z}(\mathbf{MB})$$

$$\operatorname{div}\mathbf{B} = 0$$

$$\boxed{\mathbf{F} = \operatorname{grad}(\mathbf{MB})}$$

$$F_z = M_x\left(\frac{\partial B_x}{\partial z}\right) + M_y\left(\frac{\partial B_y}{\partial z}\right) + M_z\left(\frac{\partial B_z}{\partial z}\right)$$

$$F_z = \int_V f_z dv \qquad M_z = \int_V J_z dv$$

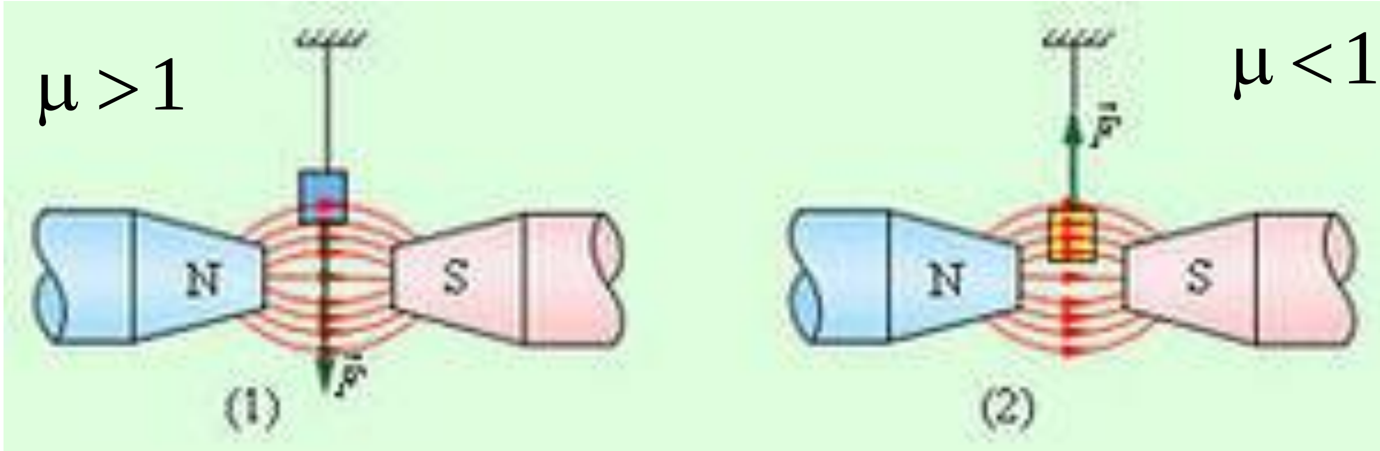
$$\int_V f_z dv = \int_V J_x\left(\frac{\partial B_x}{\partial z}\right)dv + \int_V J_y\left(\frac{\partial B_y}{\partial z}\right)dv + \int_V J_z\left(\frac{\partial B_z}{\partial z}\right)dv$$

$$f_z = J_x\left(\frac{\partial B_x}{\partial z}\right) + J_y\left(\frac{\partial B_y}{\partial z}\right) + J_z\left(\frac{\partial B_z}{\partial z}\right)$$

$$\mathbf{J} = (\mu - 1)\mathbf{B}/\mu_0\mu \qquad \mathbf{H} = \mathbf{B}/\mu\mu_0 = \mathbf{B}/\mu_0 - \mathbf{J}$$

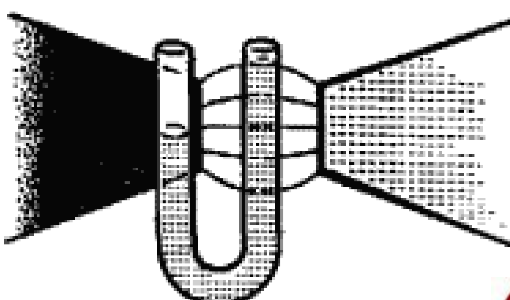
$$f_z = \frac{\mu - 1}{\mu_0\mu}\left(B_x\left(\frac{\partial B_x}{\partial z}\right) + B_y\left(\frac{\partial B_y}{\partial z}\right) + B_z\left(\frac{\partial B_z}{\partial z}\right)\right) = \frac{\mu - 1}{2\mu_0\mu}\cdot\frac{\partial}{\partial z}\mathbf{B}^2$$

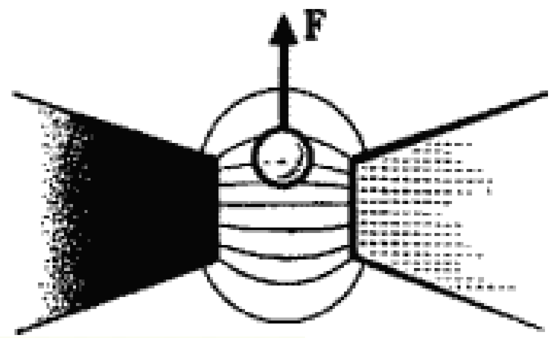
$$\mathbf{f} = \frac{\mu - 1}{2\mu_0\mu}\cdot \mathrm{grad}\left(\mathbf{B}^2\right)$$



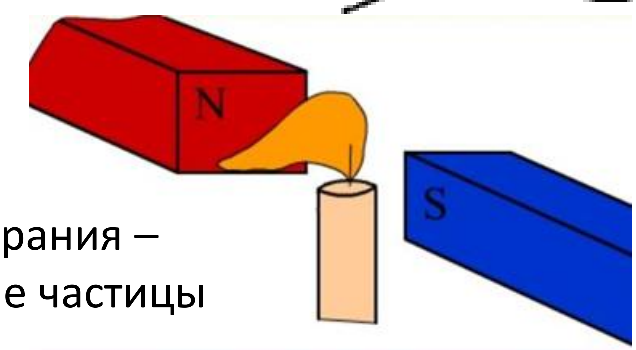
$$\mathbf{f} = \frac{\mu - 1}{2\mu_0\mu} \cdot \mathrm{grad}\left(\mathbf{B}^2\right)$$

$\mu > 1$





$\mu < 1$



Продукты сгорания –
диамагнитные частицы

Момент силы действующей на элементарный ток

$$\mathbf{N} = \int_V [\mathbf{r} \times [\mathbf{j} \times \mathbf{B}]] dv \qquad \mathbf{B} = \mathbf{B}(0) = \mathbf{B}_0$$

$$\mathbf{N} = \int_V [\mathbf{r} \times [\mathbf{j} \times \mathbf{B}]] dv = \int_V \mathbf{j}(\mathbf{r}\mathbf{B}_0) dv - \mathbf{B}_0 \int_V \mathbf{j}\mathbf{r} dv$$

$$= 0$$

$$\left(\int_V j_x x dv = \int_V j_y y dv = \int_V j_z z dv = 0 \right)$$

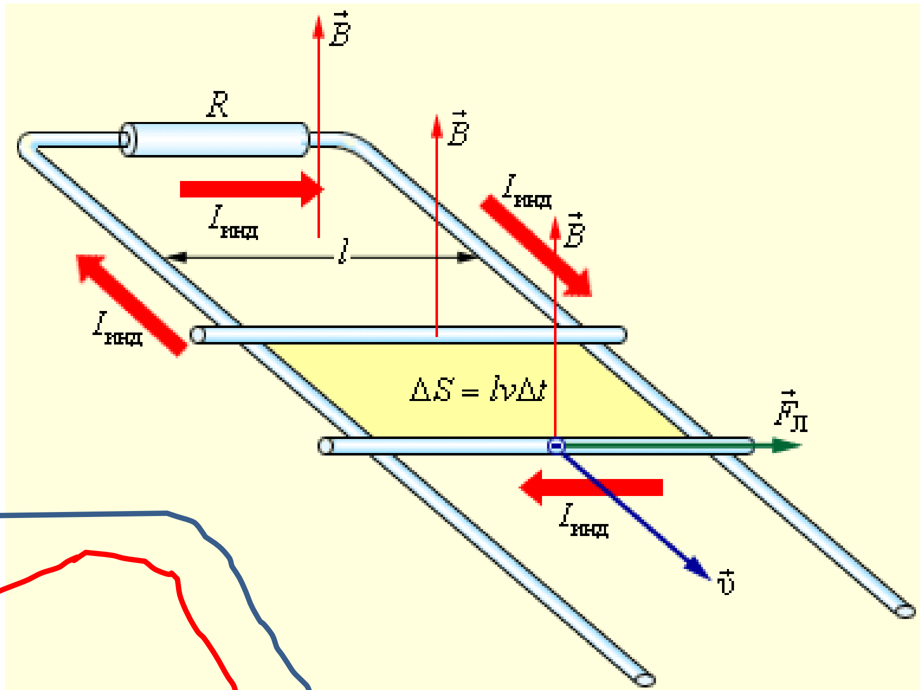
$$= 0$$

$$N_x = \int_V j_x (x B_{0x} + y B_{0y} + z B_{0z}) dv = B_{0x} \int_V j_x x dv +$$

$$+ B_{0y} \int_V j_x y dv + B_{0z} \int_V j_x z dv = -B_{0y} M_z + B_{0z} M_y$$

$$\mathbf{N} = [\mathbf{M} \times \mathbf{B}]$$

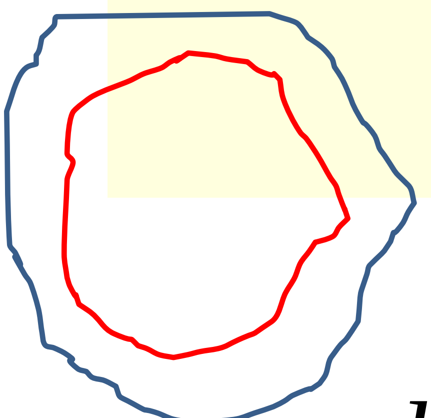
Электрическое поле в проводнике движущемся в стационарном магнитном поле



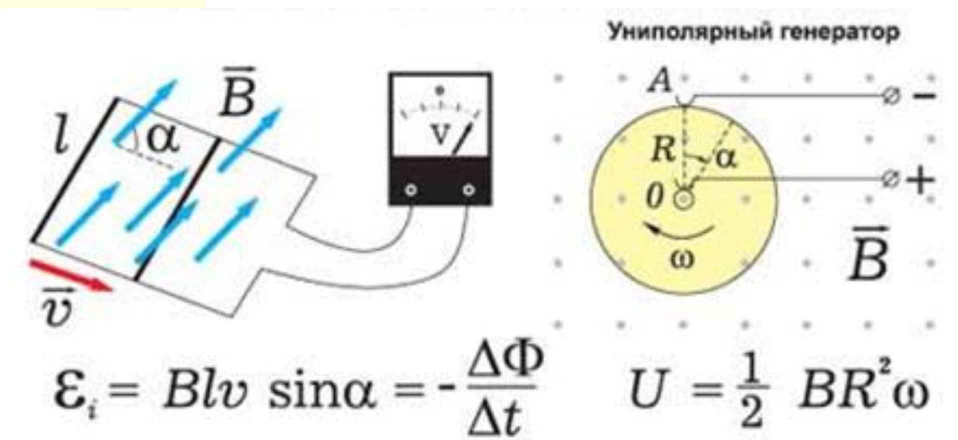
$$\mathbf{F} = q[\mathbf{v} \times \mathbf{B}]$$

$$\mathbf{E} = [\mathbf{v} \times \mathbf{B}]$$

$$\varepsilon_{инд} = vBL$$

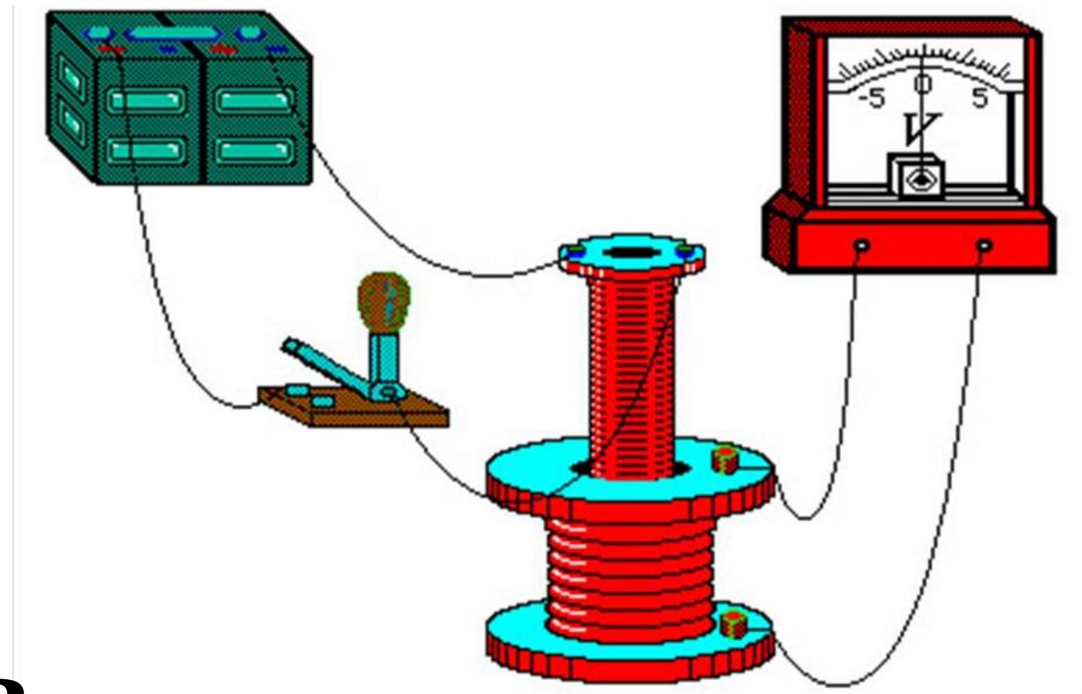


$$\varepsilon_{инд} = -\frac{d\Phi}{dt}$$

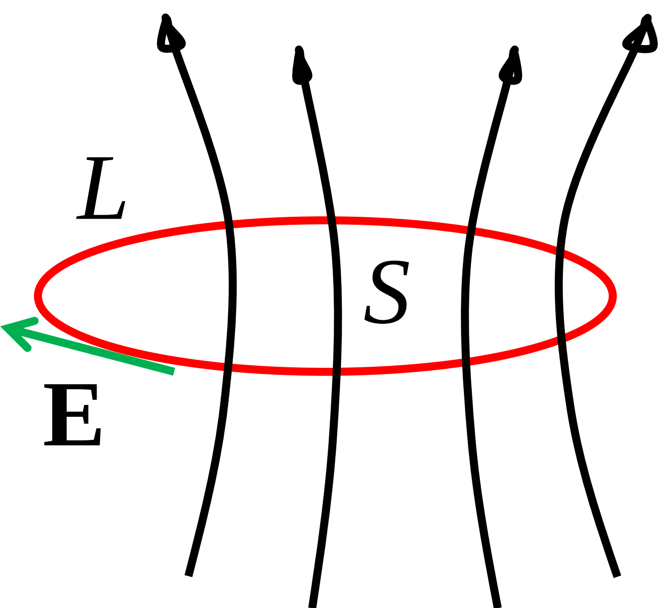


Закон электромагнитной индукции Фарадея

$$\varepsilon_{\text{инд}} = -\frac{d\Phi}{dt}$$



$$\oint_L \mathbf{E}d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

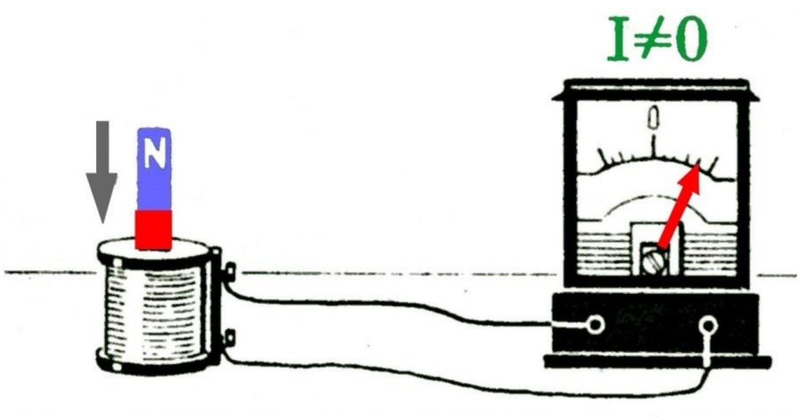


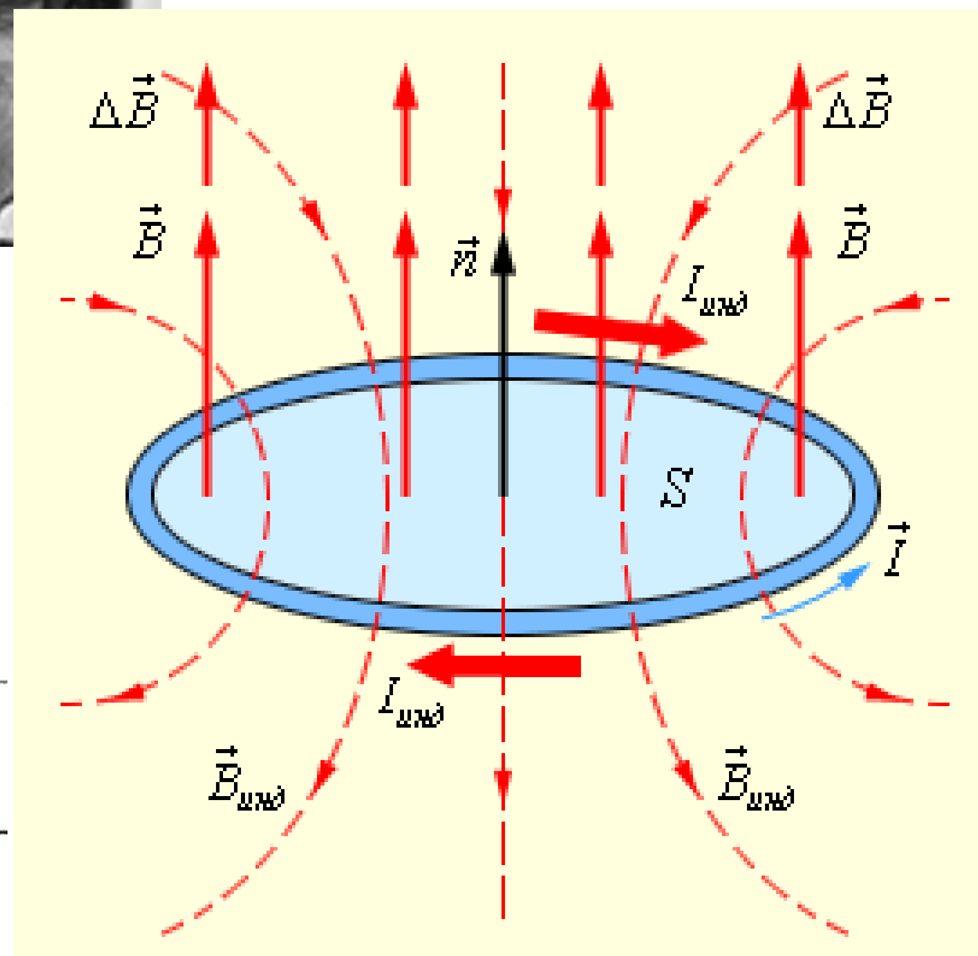
$$\frac{\partial \mathbf{B}}{\partial t} > 0$$

$$\operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\oint_L \mathbf{E}d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$



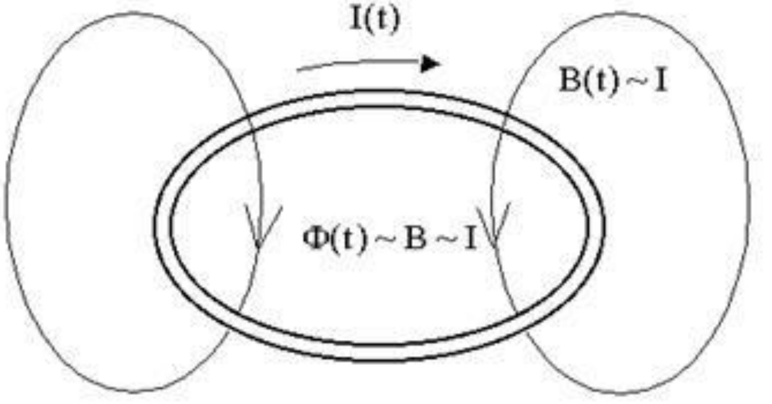


$$0 = \operatorname{div}\operatorname{rot}\mathbf{E} = -\operatorname{div}\frac{\partial \mathbf{B}}{\partial t} = -\frac{\partial}{\partial t}\operatorname{div}\mathbf{B}$$

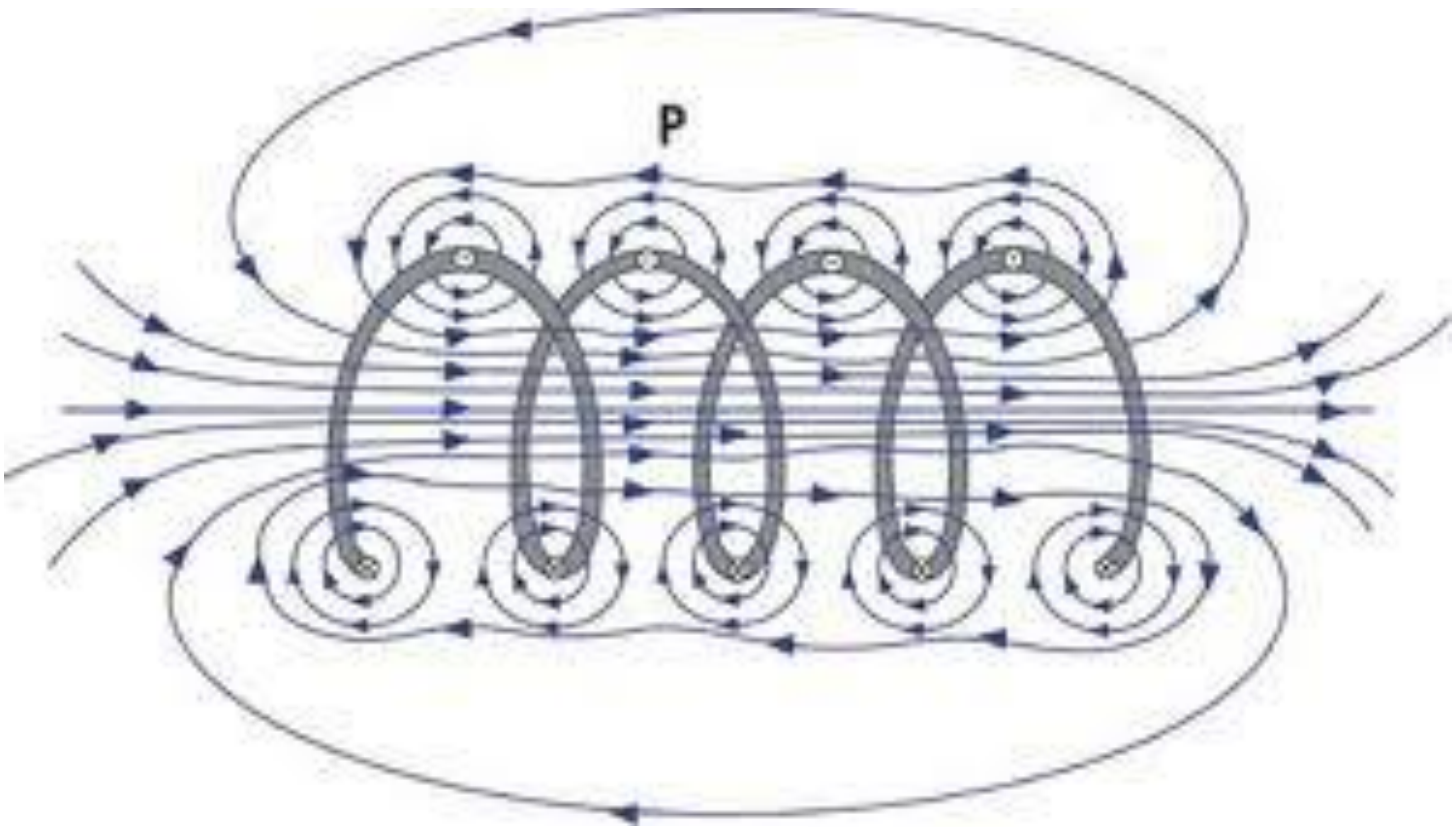
$$\operatorname{div}\mathbf{B} = \mathrm{const} \quad \Rightarrow \quad \operatorname{div}\mathbf{B} = 0$$

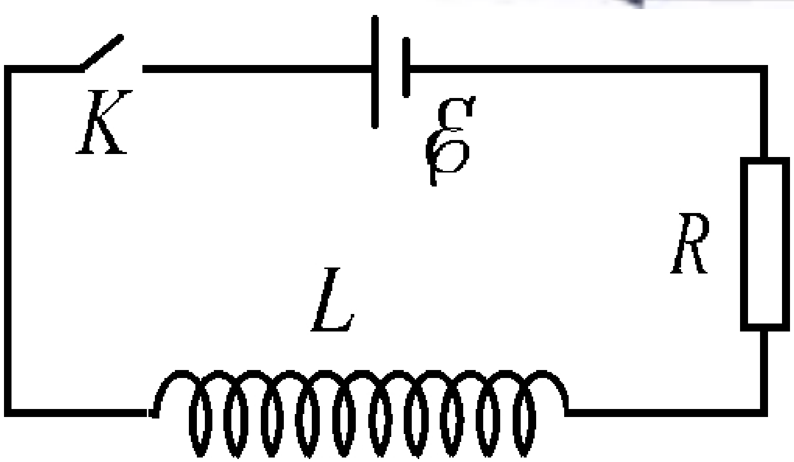
Явление самоиндукции

Джозеф Генри



$$\varepsilon_{\text{инд}} = -L\frac{dI}{dt}$$

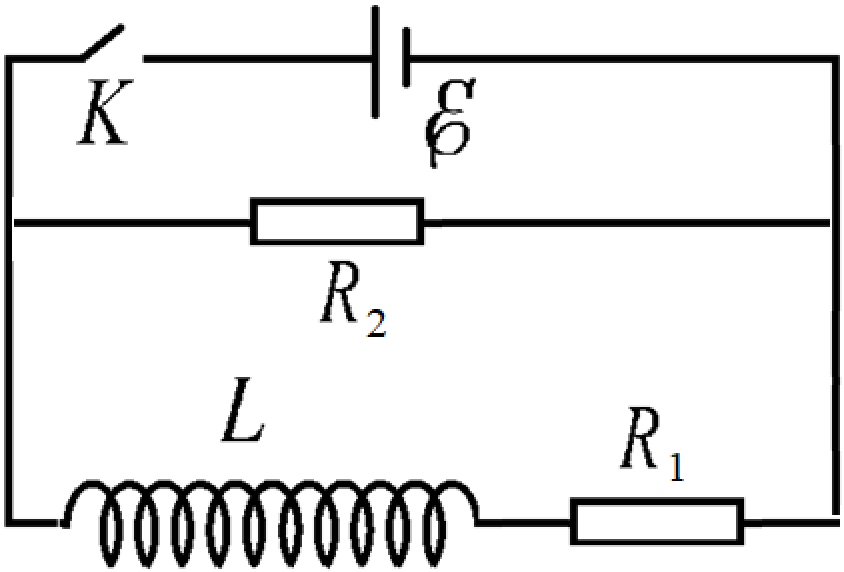




$$\varepsilon - L\frac{dI}{dt} = IR$$

$$I(0) = 0$$

$$I = \frac{\varepsilon}{R}\left(1 - \exp(-Rt/L)\right)$$



$$R_2 >> R_1$$

$$W_2 = \frac{\varepsilon^2}{R_2}\tau$$

$$-L\frac{dI}{dt} = I(R_1 + R_2)$$

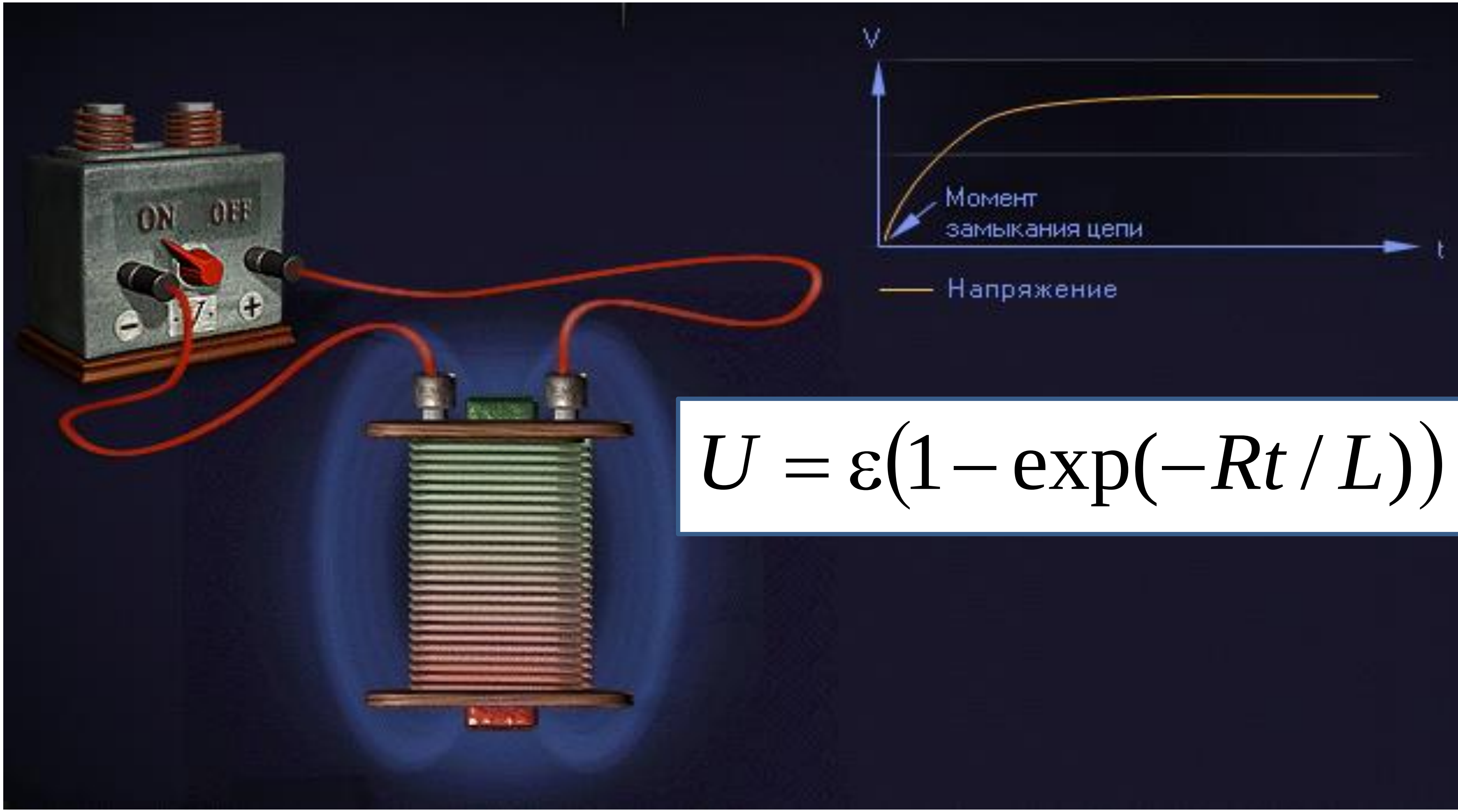
$$I(0) = \varepsilon / R_1$$

$$-L\frac{dI}{dt} = I(R_1 + R_2) \qquad \tau = \frac{R_2}{L}$$

$$I(t) = \frac{\varepsilon}{R_1}\exp[-Lt/(R_1 + R_2)] \approx \frac{\varepsilon}{R_1}\exp(-Lt/R_2)$$

$$W_2' = LI^2(0)/2 = \frac{L\varepsilon^2}{2R_1^2} \qquad W_2 = \frac{\varepsilon^2}{R_2}\tau = \frac{L\varepsilon^2}{R_2^2}$$

$$W_2' >> W_2$$



$$U = \varepsilon\left(1 - \exp(-Rt / L)\right)$$

$$A(T) = \int_0^T I\varepsilon dt = \frac{\varepsilon^2}{R}\int_0^T \left(1 - \exp(-Rt / L)\right)dt =$$

$$= \frac{\varepsilon^2 T}{R} - \frac{\varepsilon^2 L}{R^2}\left(1 - \exp(-RT / L)\right)$$

$$Q(T) = \int_0^T I^2(t) R dt =$$

$$I = \frac{\varepsilon}{R}\left(1 - \exp(-Rt/L)\right)$$

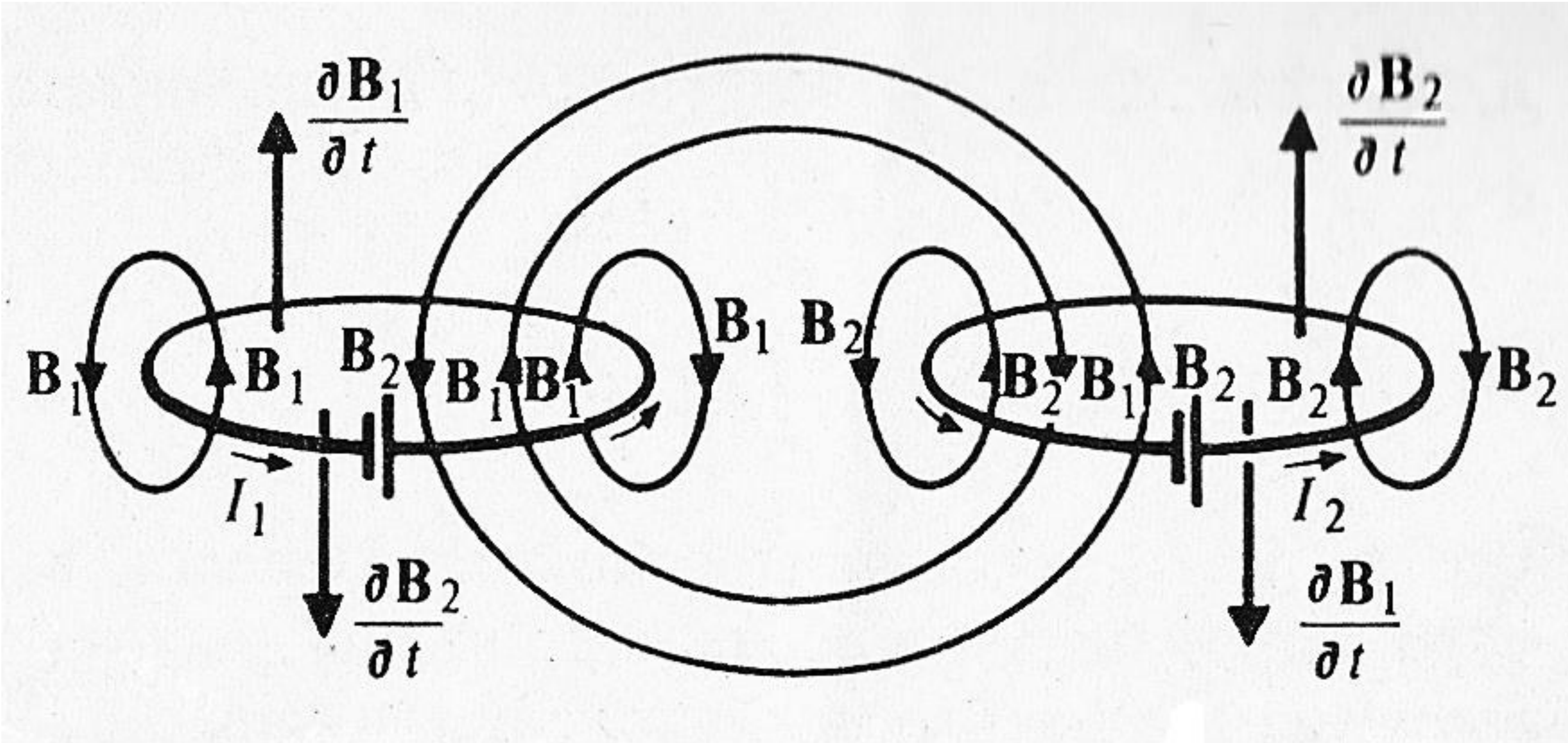
$$= \frac{\varepsilon^2}{R}\int_0^T \left(1 - 2\exp(-Rt/L) + \exp(-2Rt/L)\right) dt$$

$$= \frac{\varepsilon^2 T}{R} - \frac{2L\varepsilon^2}{R^2}\left(1 - \exp(-RT/L)\right) +$$

$$+ \frac{L\varepsilon^2}{2R^2}\left(1 - \exp(-2RT/L)\right)$$

$$W(T) = A - Q = \frac{\varepsilon^2 L}{R^2}\left(1 - \exp(-RT/L)\right) - \frac{L\varepsilon^2}{2R^2}\left(1 - \exp(-2RT/L)\right) =$$

$$= \frac{L\varepsilon^2}{2R^2}[1 - 2\exp(-RT/L) + \exp(-2RT/L)] = \frac{LI^2(T)}{2}$$



$$\varepsilon_1 - \frac{d\Phi_{11}}{dt} - \frac{d\Phi_{12}}{dt} = \varepsilon_1 - L_{11}\frac{dI_1}{dt} - L_{12}\frac{dI_2}{dt} = I_1 R_1$$

$$\varepsilon_2 - \frac{d\Phi_{21}}{dt} - \frac{d\Phi_{22}}{dt} = \varepsilon_2 - L_{21}\frac{dI_1}{dt} - L_{22}\frac{dI_2}{dt} = I_2 R_2$$

Энергия магнитного поля

$$W = \frac{1}{2}\sum_{\substack{i=1\\ j=1}}^{N} L_{ij} I_i I_j$$

Докажем, что $L_{ij} = L_{ji}$

$$\Phi_{21} = \int_{S_2} \mathbf{B}_1 d\mathbf{S}_2 = \int_{S_2} \mathrm{rot}\mathbf{A}_1 d\mathbf{S}_2 = \int_{\tilde{L}_2} \mathbf{A}_1 d\mathbf{l}_2 = \frac{\mu\mu_0 I_1}{4\pi} \int_{\tilde{L}_2} \int_{\tilde{L}_1} \frac{d\mathbf{l}_1 d\mathbf{l}_2}{r_{12}} = L_{21} I_1$$

$$L_{21} = \frac{\mu\mu_0}{4\pi} \int_{\tilde{L}_2} \int_{\tilde{L}_1} \frac{d\mathbf{l}_1 d\mathbf{l}_2}{r_{12}} = \frac{\mu\mu_0}{4\pi} \int_{\tilde{L}_1} \int_{\tilde{L}_2} \frac{d\mathbf{l}_2 d\mathbf{l}_1}{r_{21}} = L_{12}$$

$$W = \frac{1}{2}\sum_{\substack{i=1\\ j=1}}^{N} L_{ij} I_i I_j \quad \Longrightarrow \quad W = \int_V w\,dv \qquad w = \frac{\mathbf{BH}}{2}$$

$$\varepsilon = -\frac{d\Phi}{dt} = -L\frac{dI}{dt} \Longrightarrow \Phi = LI$$

$$W = LI^2/2 = I\Phi/2 \qquad \Phi = \int_S \mathbf{B}d\mathbf{S} = \int_S \mathrm{rot}\,\mathbf{A}d\mathbf{S} = \int_{\tilde{L}} \mathbf{A}d\mathbf{l}$$

$$W = \frac{I}{2}\int_{\tilde{L}} \mathbf{A}d\mathbf{l} = \frac{1}{2}\int_V \mathbf{A}\cdot\mathbf{j}dV$$

$$\mathrm{div}[\mathbf{A}\times\mathbf{H}] = \mathbf{H}\,\mathrm{rot}\,\mathbf{A} - \mathbf{A}\,\mathrm{rot}\,\mathbf{H}$$

$$\mathbf{B} = \mathrm{rot}\,\mathbf{A} \qquad \mathbf{j} = \mathrm{rot}\,\mathbf{H}$$

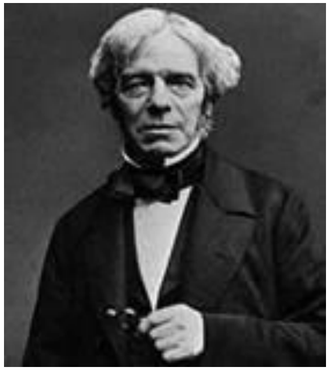
$$\mathbf{A}\cdot\mathbf{j} = \mathbf{H}\cdot\mathbf{B} - \mathrm{div}[\mathbf{A}\times\mathbf{H}]$$

$$W = \frac{1}{2}\int_V \mathbf{A}\cdot\mathbf{j}dV = \frac{1}{2}\int_V \mathbf{H}\cdot\mathbf{B}dV - \cancel{\frac{1}{2}\int_V \mathrm{div}[\mathbf{A}\times\mathbf{H}]dV} = 0$$

$$\boxed{W = \int_V w dv \qquad w = \frac{\mathbf{BH}}{2}}$$

$$\int_V \mathrm{div}[\mathbf{A}\times\mathbf{B}]dV = \int_S [\mathbf{A}\times\mathbf{B}]d\mathbf{S} \to 0$$

Нестационарное электромагнитное поле в вакууме

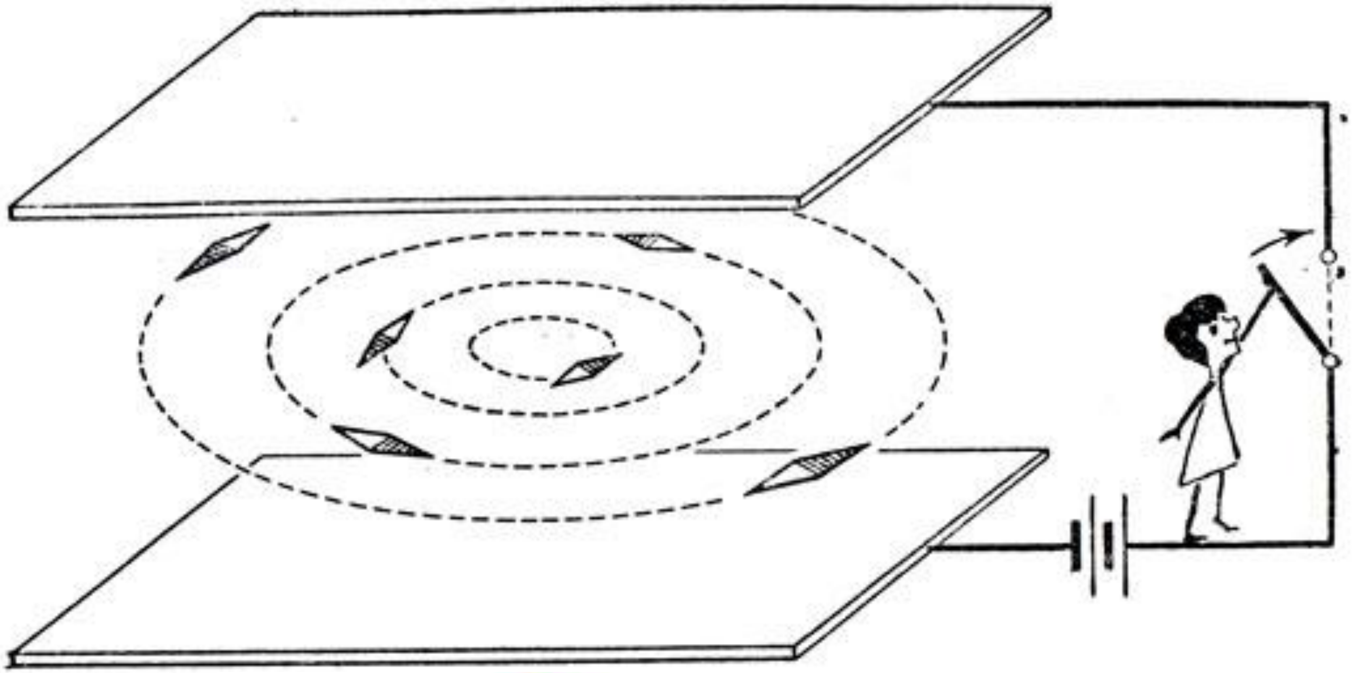


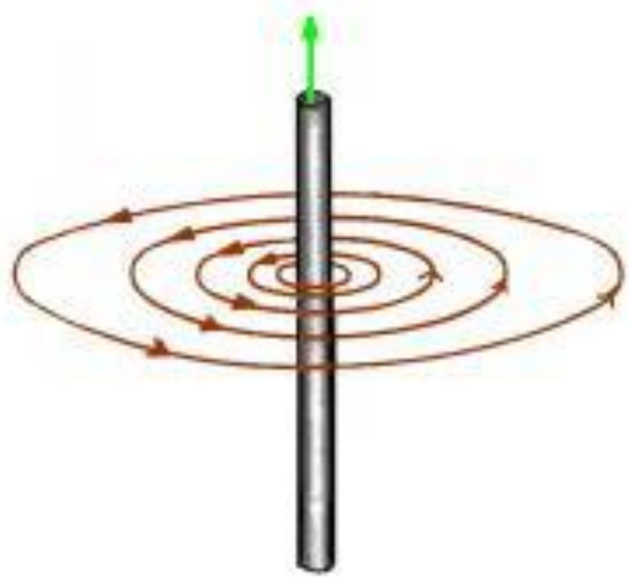
$$\oint_L \mathbf{E} d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \qquad \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad \operatorname{div} \mathbf{B} = 0$$

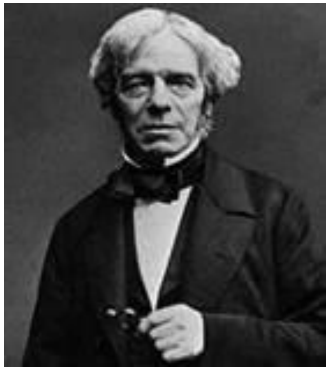
$$\oint_L \mathbf{B} d\mathbf{l} = \mu_0 \int_S \mathbf{j} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

$$\operatorname{rot} \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$





Нестационарное электромагнитное поле в вакууме



$$\oint_L \mathbf{E}d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \Rightarrow \operatorname{rot} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \Rightarrow \operatorname{div} \mathbf{B} = 0$$

$$\oint_L \mathbf{B}d\mathbf{l} = \mu_0 \int_S \mathbf{j} \cdot d\mathbf{S} + \mu_0 \varepsilon_0 \int_S \frac{\partial \mathbf{E}}{\partial t} \cdot d\mathbf{S}$$

$$\Downarrow$$

$$\operatorname{rot} \mathbf{B} = \mu_0 \mathbf{j} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\operatorname{rot}\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$0 = \operatorname{div}\operatorname{rot}\mathbf{B}(\mathbf{r},t) = \mu_0\operatorname{div}\mathbf{j} + \mu_0\varepsilon_0\operatorname{div}\frac{\partial\mathbf{E}(\mathbf{r},t)}{\partial t} = -\mu_0\frac{\partial\rho}{\partial t} + \mu_0\varepsilon_0\operatorname{div}\frac{\partial\mathbf{E}}{\partial t} =$$

$$= \mu_0\frac{\partial}{\partial t}\left(-\rho + \varepsilon_0\operatorname{div}\mathbf{E}\right) \Rightarrow \operatorname{div}\mathbf{E}(\mathbf{r},t) = \rho/\varepsilon_0 + f(\mathbf{r})$$

$$f(\mathbf{r}) = 0 \qquad \operatorname{div}\mathbf{E} = \rho/\varepsilon_0 \Rightarrow \int_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho\, dv$$

$$\operatorname{rot}\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}$$

$$\operatorname{div}\mathbf{B} = 0$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$\operatorname{div}\mathbf{E} = \rho/\varepsilon_0$$

1.Восемь уравнений, но шесть неизвестных (второе и четвертое уравнения получаются из первого и третьего с помощью уравнения непрерывности.

2.Принцип суперпозиции электромагнитных полей.

3.Для решения нужны начальные и граничные условия

Уравнения Максвелла в среде

$$\mathrm{rot}\,\mathbf{B}_m = \mu_0 \overline{\rho \mathbf{v}} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}_m}{\partial t} \Rightarrow \ \mathbf{?}$$

$$\mathrm{div}\,\mathbf{B}_m = 0 \Rightarrow \mathrm{div}\,\mathbf{B} = 0$$

$$\mathrm{rot}\,\mathbf{E}_m = -\frac{\partial \mathbf{B}_m}{\partial t} \Rightarrow \mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathrm{div}\,\mathbf{E}_m = \overline{\rho} / \varepsilon_0 \Rightarrow \mathrm{div}\,\mathbf{D} = \rho_{своб}$$

$$\overline{f} = \frac{1}{2\tau} \int_{-\tau}^{\tau} \frac{1}{v_0} \int_{v_0} f_m \, dv\, dt$$

$$\overline{\rho} = \rho_{своб} + \rho_{связ} = \rho_{своб} - \mathrm{div}\,\mathbf{P}$$

$$\overline{\rho \mathbf{v}} = \mathbf{j}_{пров} + \mathbf{j}_{мол} + \gamma \partial \mathbf{E} / \partial t = \mathbf{j} + \mathrm{rot}\,\mathbf{J} + \gamma \partial \mathbf{E} / \partial t \qquad (\mathbf{j} \propto \mathbf{B})$$

$$0=\frac{\partial(\rho_{своб}+\rho_{связ})}{\partial t}+\mathrm{div}(\mathbf{j}_{пров}+\mathbf{j}_{мол}+\gamma\partial\mathbf{E}/\partial t)=$$

$$=\underline{\frac{\partial\rho_{своб}}{\partial t}}+\underline{\mathrm{div}\,\mathbf{j}_{пров}}+\frac{\partial(-\mathrm{div}\,\mathbf{P})}{\partial t}+\mathrm{div}\,\mathrm{rot}\,\mathbf{J}+\mathrm{div}\,\gamma\partial\mathbf{E}/\partial t=$$

$$=0$$

$$=\frac{\partial\,\mathrm{div}(-\mathbf{P}+\gamma\mathbf{E})}{\partial t}\qquad\Rightarrow\qquad\mathbf{P}=\gamma\mathbf{E}$$

$$\mathrm{rot}\,\mathbf{B}_m=\underline{\mu_0\overline{\rho\mathbf{v}}}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}_m}{\partial t}$$

$$\gamma\partial\mathbf{E}/\partial t$$

$$\mathrm{rot}\,\mathbf{B}_m=\underline{\mu_0(\mathbf{j}+\mathrm{rot}\,\mathbf{J}+\partial\mathbf{P}/\partial t)}+\mu_0\varepsilon_0\frac{\partial\mathbf{E}_m}{\partial t}$$

$$\mathbf{H}=\mathbf{B}/\mu_0-\mathbf{J}\qquad\qquad\mathbf{D}=\varepsilon_0\mathbf{E}+\mathbf{P}$$

$$\mathrm{rot}\,\mathbf{H}=\mathbf{j}+\frac{\partial\mathbf{P}}{\partial t}+\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}=\mathbf{j}+\frac{\partial\mathbf{D}}{\partial t}$$

Уравнения Максвелла в среде

$$\operatorname{rot}\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \qquad \oint_L \mathbf{H}d\mathbf{l} = \int_S \mathbf{j}d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} d\mathbf{S}$$

$$\operatorname{div}\mathbf{B} = 0 \qquad \oint_S \mathbf{B}d\mathbf{S} = 0$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \oint_L \mathbf{E}d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t} d\mathbf{S}$$

$$\operatorname{div}\mathbf{D} = \rho \qquad \oint_S \mathbf{D}d\mathbf{S} = \int_V \rho dv$$

Материальные уравнения

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E} \qquad \mathbf{B} = \mu\mu_0\mathbf{H}$$

Уравнения Максвелла в среде

$$\operatorname{rot}\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\operatorname{div}\mathbf{B} = 0$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{div}\mathbf{D} = \rho$$

$$\mathbf{D} = \varepsilon\varepsilon_0\mathbf{E}$$

$$\mathbf{B} = \mu\mu_0\mathbf{H}$$

1. Заряженные тела и провода с токами неподвижны.
2. Материальные константы не зависят от времени.
3. Отсутствуют постоянные магниты и ферромагнетики.
4. При заданные граничных и начальных условиях ее решение единственно.

Граничные условия

$$\mathbf{D}_{n2} - \mathbf{D}_{n1} = 0 \qquad \mathbf{B}_{n2} - \mathbf{B}_{n1} = 0$$

$$\mathbf{E}_{t2} - \mathbf{E}_{t1} = 0 \qquad \mathbf{H}_{t2} - \mathbf{H}_{t1} = 0$$

Единственность решения

$$\mathbf{D}_2 - \mathbf{D}_1 = \mathbf{D} \qquad \mathbf{B}_2 - \mathbf{B}_1 = \mathbf{B} \qquad \mathbf{E}_2 - \mathbf{E}_1 = \mathbf{E} \qquad \mathbf{H}_2 - \mathbf{H}_1 = \mathbf{H}$$

$$\operatorname{rot}\mathbf{H} = \sigma\mathbf{E} + \frac{\partial \mathbf{D}}{\partial t}, \quad \operatorname{div}\mathbf{B} = 0, \quad \operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \ \operatorname{div}\mathbf{D} = 0$$

В начальный момент времени $\mathbf{D} = \mathbf{E} = \mathbf{B} = \mathbf{H} = \mathbf{0}$

$\mathbf{D}$, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ на границе равны нулю

$$\mathbf{a}\,\mathrm{rot}\,\mathbf{b} = \mathbf{b}\,\mathrm{rot}\,\mathbf{a} - \mathrm{div}[\mathbf{a}\times\mathbf{b}] \qquad \mathbf{b} = \mathbf{H} \qquad \mathbf{a} = \mathbf{E}$$

$$\int_V \sigma \mathbf{E}^2 dv = \int_V \mathbf{E}\left(\mathrm{rot}\,\mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}\right) dv = \int_V \mathbf{H}\,\mathrm{rot}\,\mathbf{E}\,dv - \int_V \mathrm{div}[\mathbf{E}\times\mathbf{H}]\,dv -$$

$$-\varepsilon\varepsilon_0 \int_V \mathbf{E}\frac{\partial \mathbf{E}}{\partial t} dv = -(\mu\mu_0)^{-1}\int_V \mathbf{B}\frac{\partial \mathbf{B}}{\partial t} dv - \int_S [\mathbf{E}\times\mathbf{H}]\,d\mathbf{S} - \varepsilon\varepsilon_0 \int_V \mathbf{E}\frac{\partial \mathbf{E}}{\partial t} dv =$$

$$= 0$$

Нулевые граничные условия

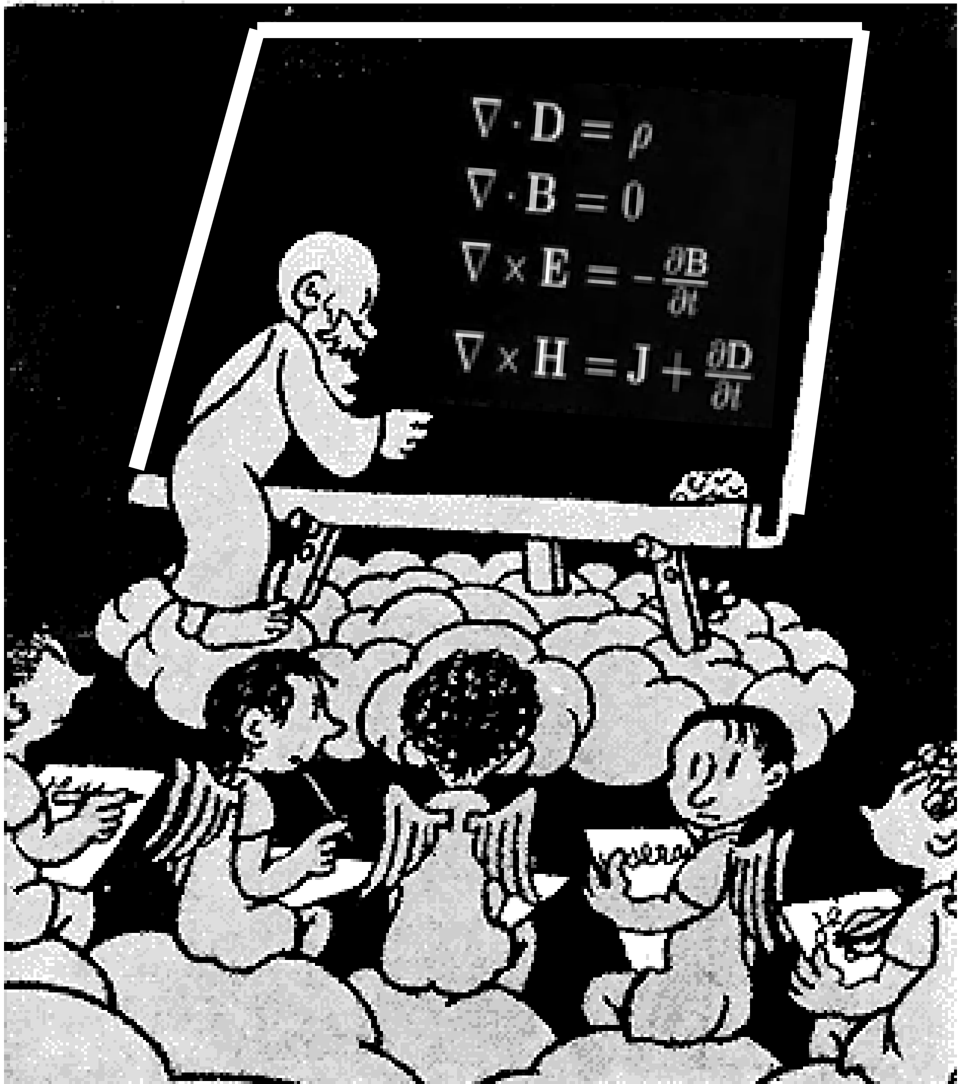
$$= -\frac{\partial}{\partial t}\left(\int_V \left(\frac{1}{2}\varepsilon\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu\mu_0}\mathbf{B}^2\right)\right) dv$$

$$\frac{\partial}{\partial t}\left(\int_V \left(\frac{1}{2}\varepsilon\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu\mu_0}\mathbf{B}^2\right) dv\right) = -\int_V \sigma \mathbf{E}^2 dv \le 0$$

С ростом времени $\int_V \left(\frac{1}{2}\varepsilon\varepsilon_0 \mathbf{E}^2 + \frac{1}{2\mu\mu_0}\mathbf{B}^2\right) dv$ уменьшается, но из-за нулевых

начальных условий $\left(\int_V \left(\frac{1}{2}\varepsilon\varepsilon_0 \mathbf{E}^2 + \frac{1}{2}\mu\mu_0 \mathbf{B}^2\right) dV\right)_{t=0} = 0$

$$\Rightarrow \quad \mathbf{D} = 0, \quad \mathbf{E} = 0, \quad \mathbf{B} = 0, \quad \mathbf{H} = 0$$



AND GOD SAID

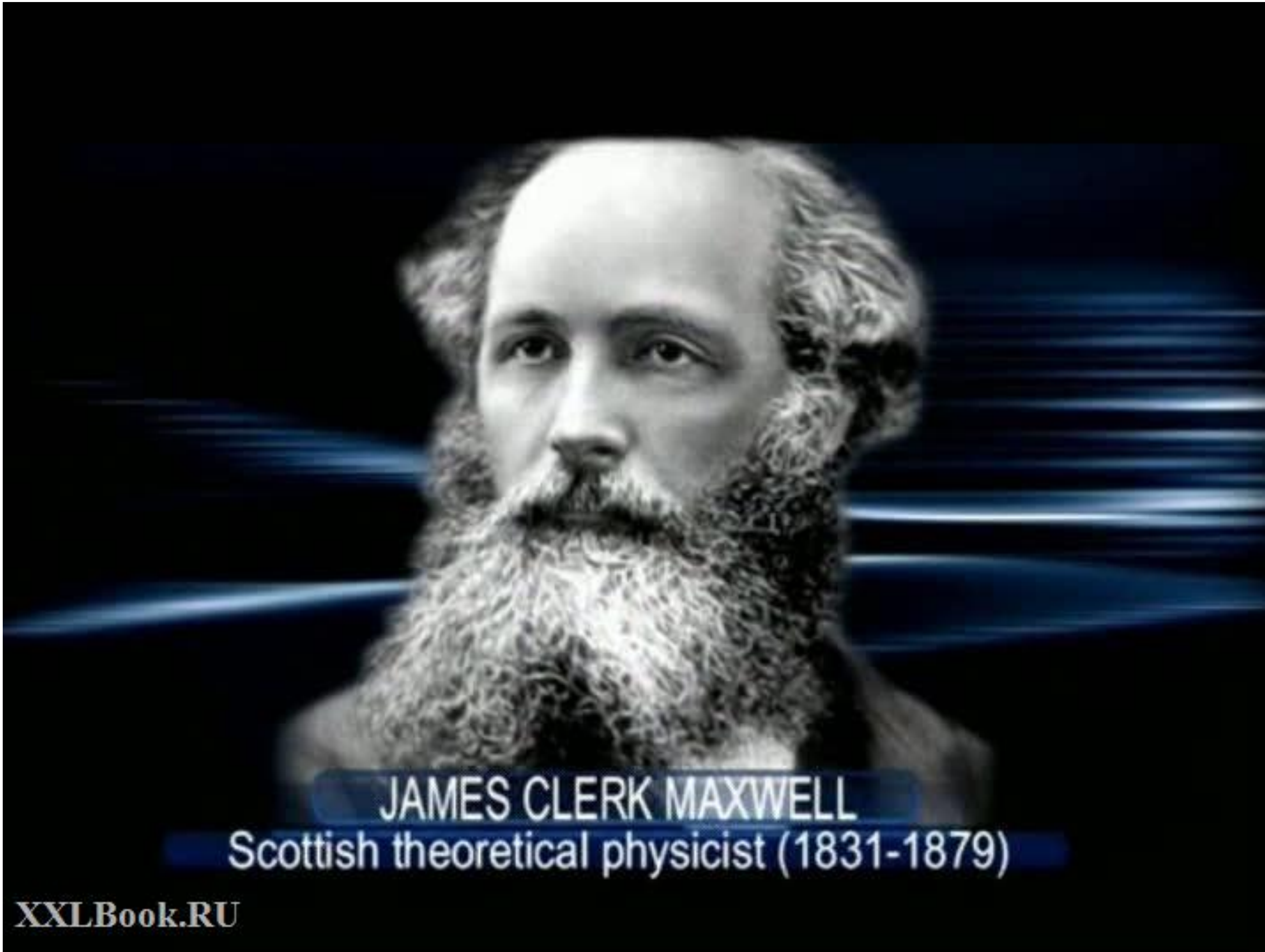
$$\mathrm{div}\,\mathbf{D} = \rho$$

$$\mathrm{div}\,\mathbf{B} = 0$$

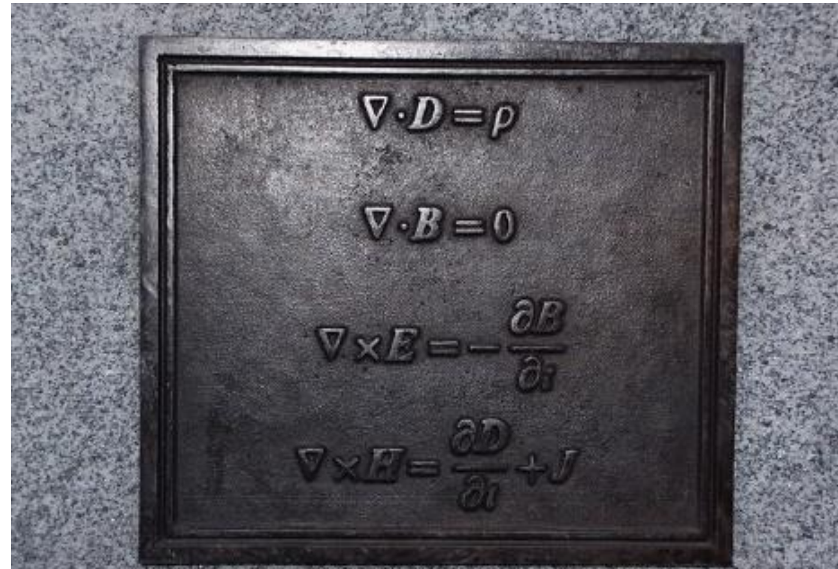
$$\mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathrm{rot}\,\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$$

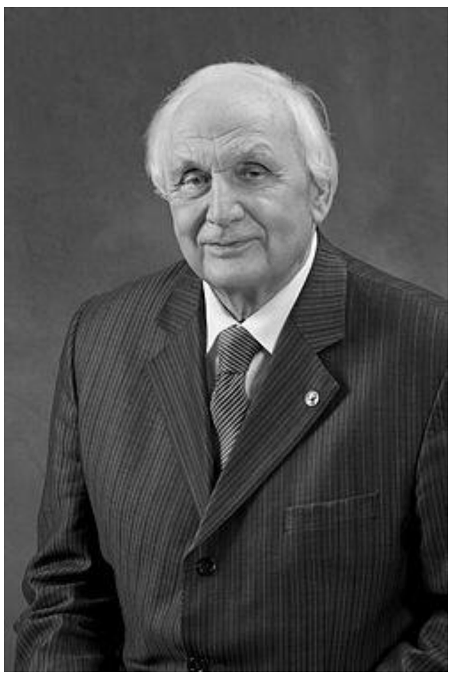
AND THEN THERE WAS LIGHT


JAMES CLERK MAXWELL
Scottish theoretical physicist (1831-1879)


$\nabla \cdot D = \rho$
$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = J + \frac{\partial D}{\partial t}$


$\nabla \cdot D = \rho$
$\nabla \cdot B = 0$
$\nabla \times E = -\frac{\partial B}{\partial t}$
$\nabla \times H = \frac{\partial D}{\partial t} + J$

«….Бог создал Максвелла, Ньютона
Им создан Пушкин и Рембрандт
По нам неведомым законам
Бог дарит избранным талант. …»



Владимир Александрович Ильин

1928-2014

Закон сохранения энергии

$$W(t)=\frac{1}{2}\oint(\mathbf{ED}+\mathbf{BH})dV$$

$$\int_V \sigma\mathbf{E}^2 dv=\int_V \mathbf{jE}dv=\int_V \mathbf{E}\left(\operatorname{rot}\mathbf{H}-\frac{\partial\mathbf{D}}{\partial t}\right)dV=\int_V \mathbf{H}\operatorname{rot}\mathbf{E}dv-$$

$$-\int_V \operatorname{div}[\mathbf{E}\times\mathbf{H}]dv-\int_V \mathbf{E}\frac{\partial\mathbf{D}}{\partial t}dv=$$

$$\mathbf{a}\operatorname{rot}\mathbf{b}=\mathbf{b}\operatorname{rot}\mathbf{a}-\operatorname{div}[\mathbf{a}\times\mathbf{b}]$$

$$\mathbf{a}=\mathbf{E} \qquad \mathbf{b}=\mathbf{H}$$

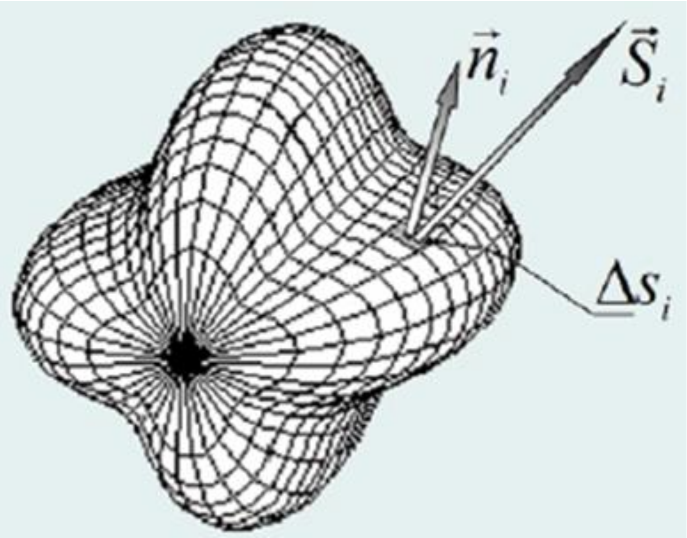
$$=-\int_V \mathbf{H}\frac{\partial\mathbf{B}}{\partial t}dv-\int_S[\mathbf{E}\times\mathbf{H}]d\mathbf{S}-\int_V \mathbf{E}\frac{\partial\mathbf{D}}{\partial t}dv \Rightarrow$$

$$\mathbf{H}\frac{\partial\mathbf{B}}{\partial t}=\frac{1}{2}\frac{\partial}{\partial t}\mathbf{BH}$$

$$\Rightarrow \int_V \mathbf{j}\cdot\mathbf{E}dv=-\left[\frac{1}{2}\int_V\left(\frac{\partial}{\partial t}\mathbf{BH}+\frac{\partial}{\partial t}\mathbf{ED}\right)dv\right]-\int_S[\mathbf{E}\times\mathbf{H}]d\mathbf{S}$$

$$\int_V \mathbf{j}\cdot\mathbf{E}dv=-\frac{\partial}{\partial t}\left(\frac{1}{2}\int_V(\mathbf{BH}+\mathbf{ED})dv\right)-\int_S[\mathbf{E}\times\mathbf{H}]d\mathbf{S}$$

$$\boxed{\frac{\partial W}{\partial t}=-\int_V \mathbf{j}\cdot\mathbf{E}dV-\int_S[\mathbf{E}\times\mathbf{H}]d\mathbf{S} \qquad \mathbf{S}_p=[\mathbf{E}\times\mathbf{H}]}$$



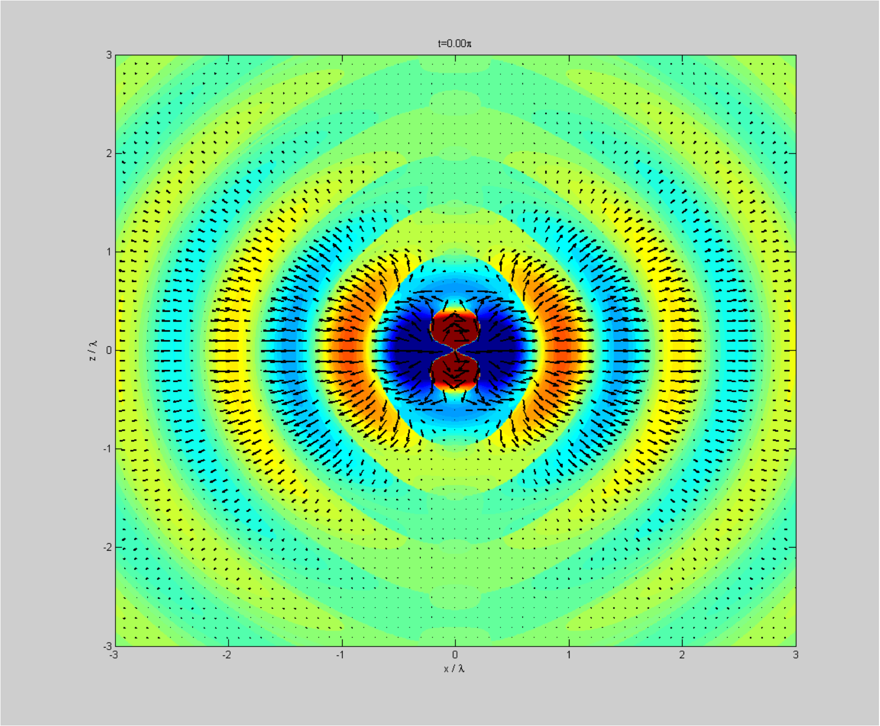
$$\mathbf{S} = [\mathbf{E} \times \mathbf{H}]$$

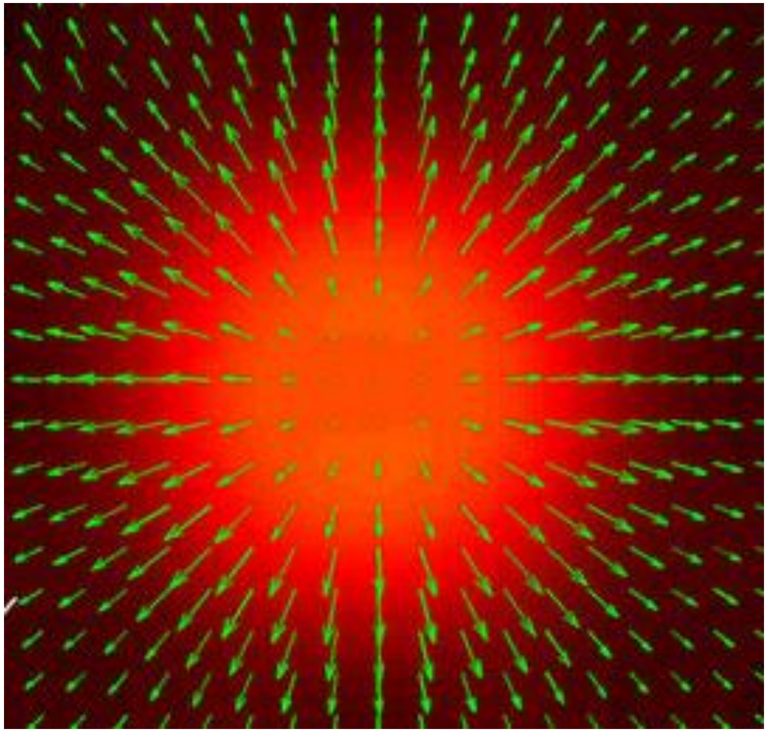
Отсутствие токов:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\int_V (\mathbf{BH} + \mathbf{ED})dV\right) + \int_S [\mathbf{E} \times \mathbf{H}]d\mathbf{S} = 0$$

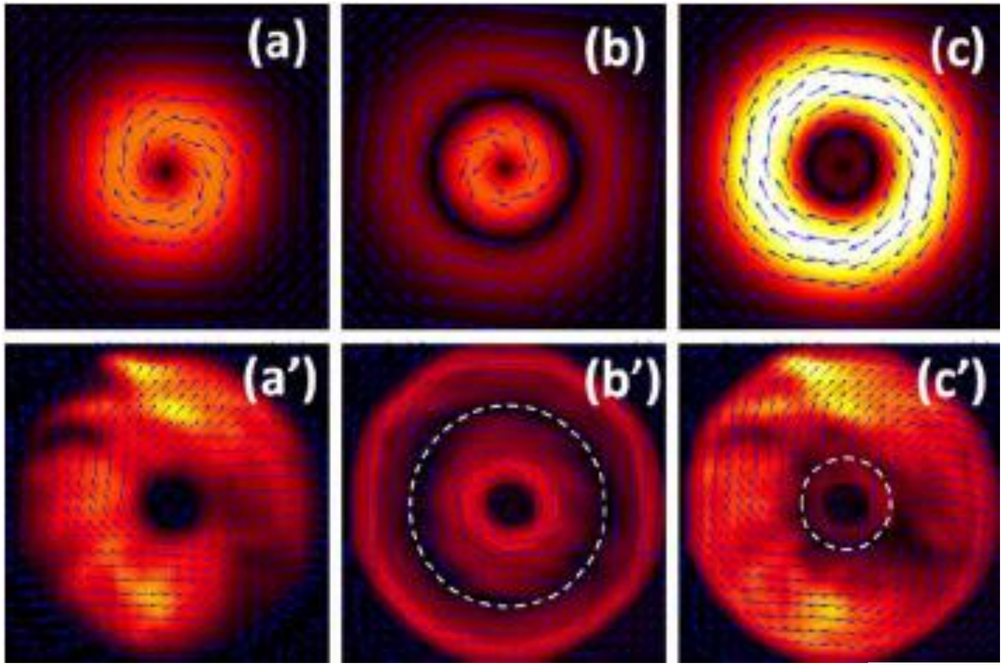
Диполь

Наночастица



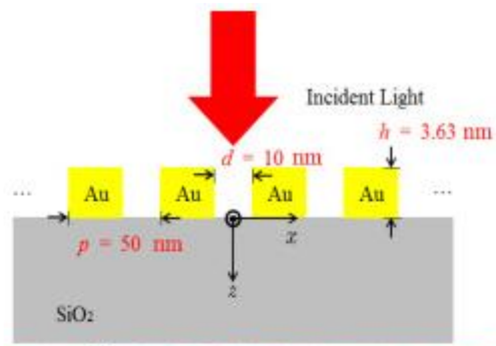


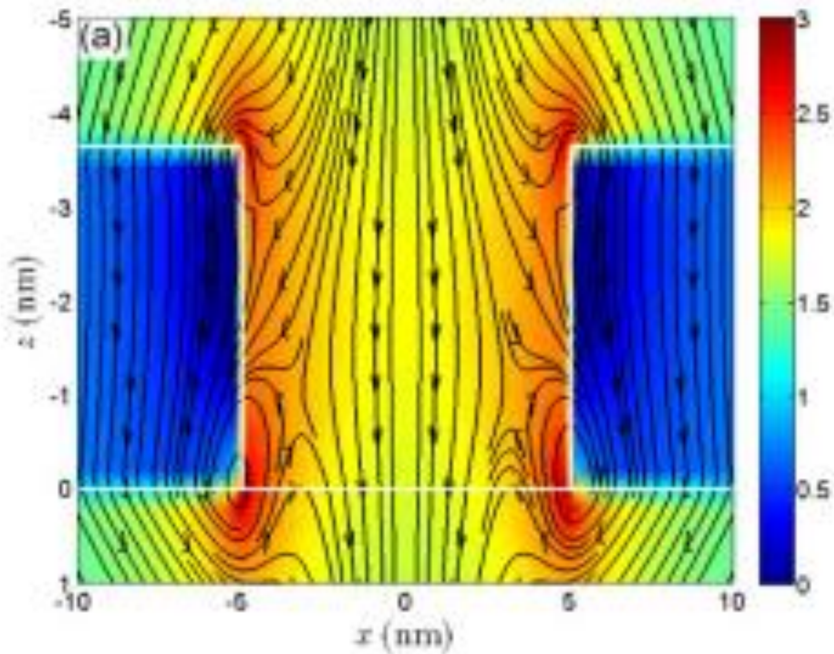
Finite-Difference Time-Domain method



Лазерный пучок

Нанорешетка на поверхности





Существование электромагнитных волн

$$\operatorname{rot}\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t} \qquad \operatorname{div}\mathbf{B} = 0$$

$$\operatorname{rot}\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \operatorname{div}\mathbf{D} = \rho = 0$$

$$\operatorname{rot}\operatorname{rot}\mathbf{E} = \operatorname{grad}\overbrace{\operatorname{div}\mathbf{E}}^{=0} - \underline{\Delta\mathbf{E}} = -\frac{\partial}{\partial t}\operatorname{rot}\mathbf{B} =$$

$$= -\frac{\partial}{\partial t}\mu\mu_0\frac{\partial \mathbf{D}}{\partial t} - \frac{\partial \mathbf{j}}{\partial t}\mu\mu_0 = -\underline{\varepsilon\varepsilon_0\mu\mu_0\frac{\partial^2 \mathbf{E}}{\partial t^2}} - \mu\mu_0\frac{\partial \mathbf{j}}{\partial t}$$

$$\Delta\mathbf{E} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu\mu_0\frac{\partial \mathbf{j}}{\partial t} \qquad \varepsilon\varepsilon_0\mu\mu_0 = \frac{\varepsilon\mu}{c^2} = \frac{1}{v^2}$$

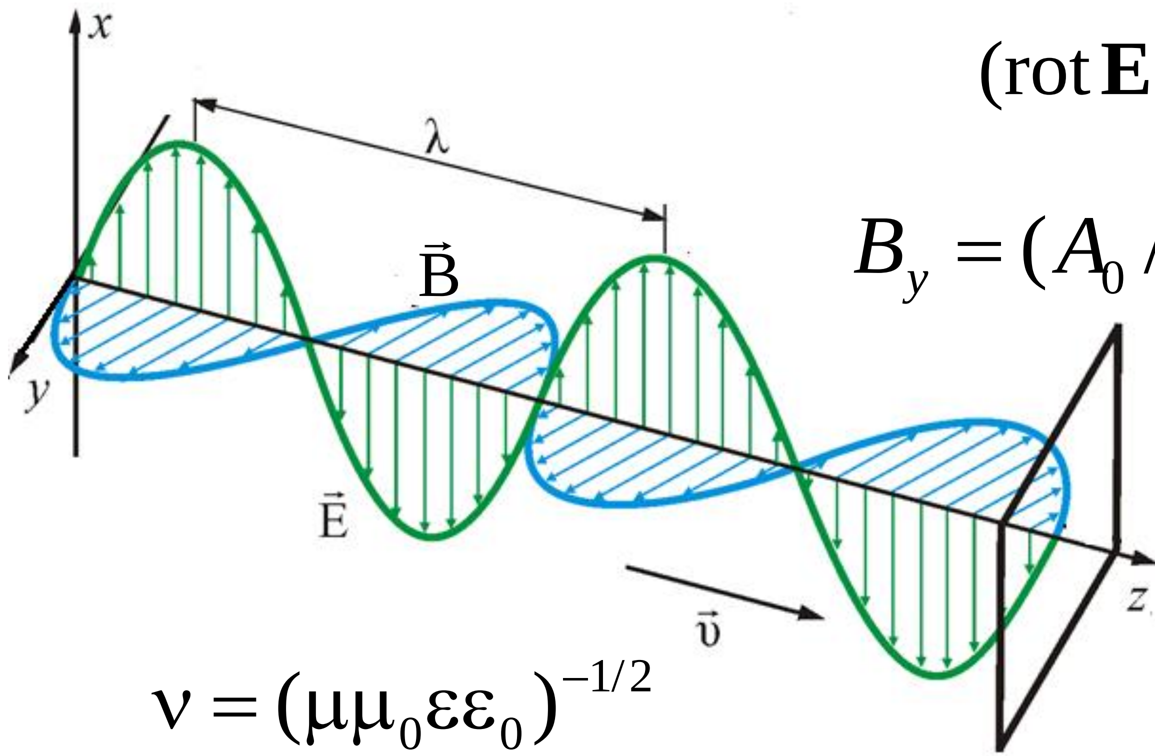
$$\frac{\partial^2 \mathbf{E}}{\partial z^2} - \frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \quad \Rightarrow \quad \mathbf{E} = \mathbf{F}_1(t - z/v) + \mathbf{F}_2(t + z/v)$$

$$\mathbf{E} = \mathbf{A}_1 \sin[\omega(t - z/v) + \varphi_1] + \mathbf{A}_2 \sin[\omega(t + z/v) + \varphi_2]$$

$$E_x = A_0 \sin[\omega(t - z/v)] \qquad \mathrm{rot}\,\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (\mathrm{rot}\,\mathbf{E})_{x,z} = 0$$

$$E_y = 0 \qquad E_z = 0$$

$$(\mathrm{rot}\,\mathbf{E})_y = \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$$



$$B_y = (A_0/v)\sin[\omega(t - z/v)]$$

$$E_x = \nu B_y$$

$$E_x = \nu\mu\mu_0 H_y$$

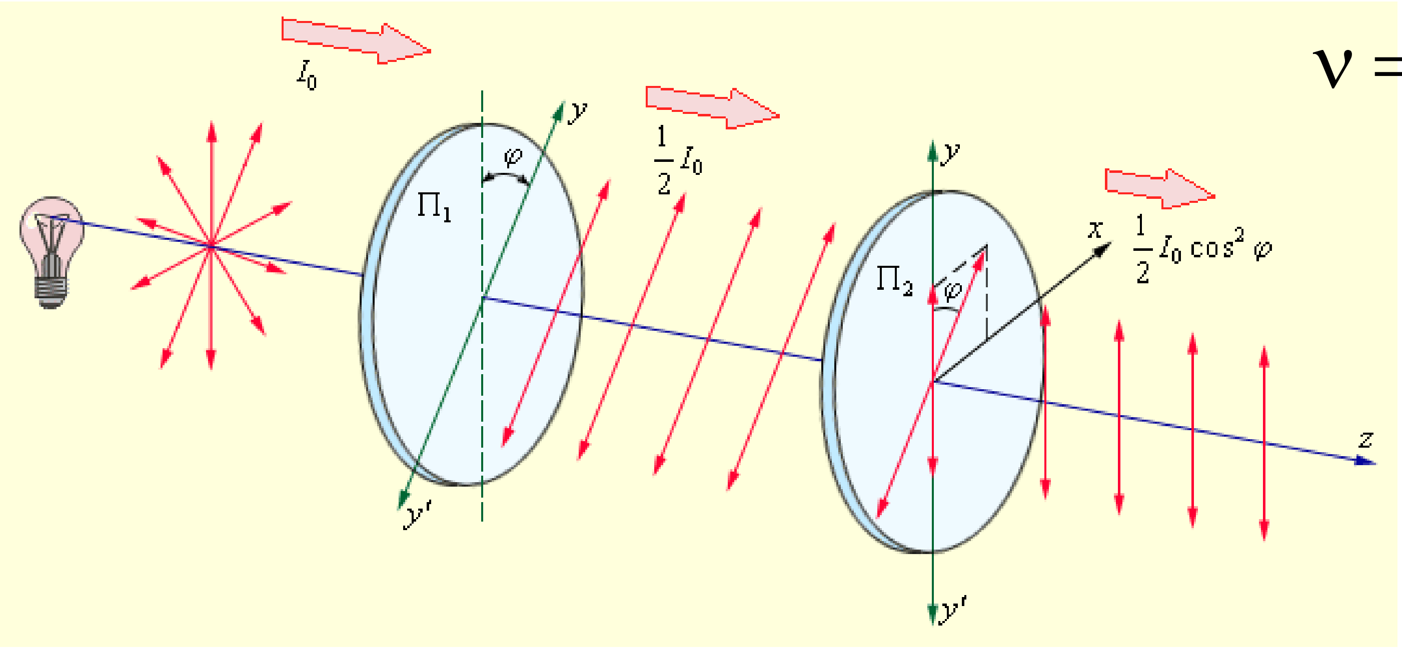
$$\nu = (\mu\mu_0\varepsilon\varepsilon_0)^{-1/2}$$

$$S = \langle |[\mathbf{E}\times\mathbf{H}]| \rangle = \langle E_x H_y \rangle = (\nu\mu\mu_0)^{-1} \langle E_x^2 \rangle =$$

$$= (\varepsilon\varepsilon_0 / \mu\mu_0)^{1/2} A_0^2 \langle \sin^2[\omega(t - z/v)] \rangle = (\varepsilon\varepsilon_0 / \mu\mu_0)^{1/2} A_0^2 / 2$$

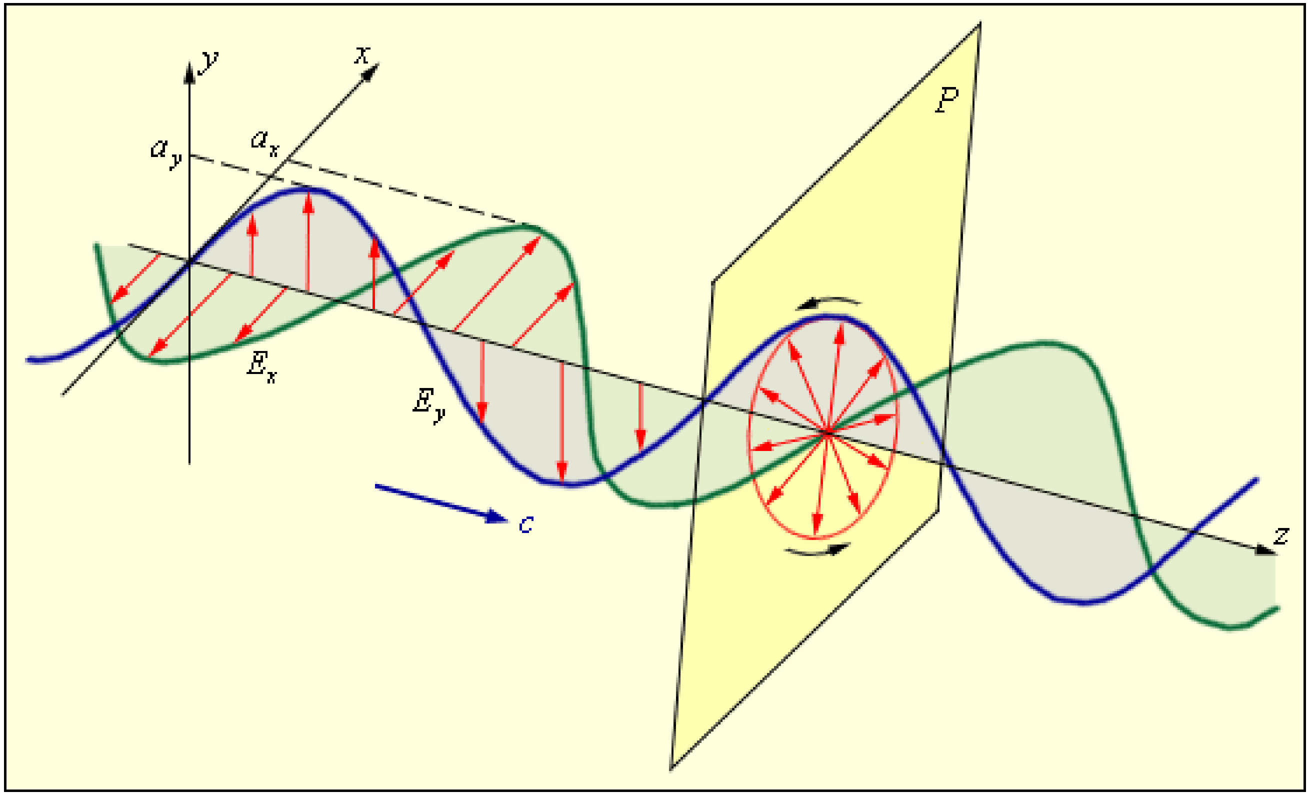
$$w = \frac{1}{2} \langle |(\mathbf{ED}+\mathbf{BH})| \rangle = \frac{\varepsilon\varepsilon_0}{2} \langle E^2 \rangle + \frac{1}{2\mu\mu_0} \langle B^2 \rangle =$$

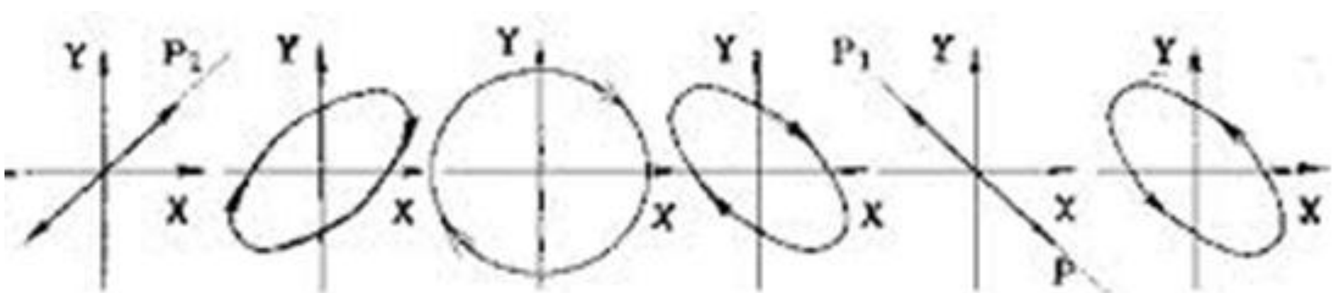
$$= \frac{\varepsilon\varepsilon_0}{2} \langle E^2 \rangle + \frac{1}{2\mu\mu_0\nu^2} \langle E^2 \rangle = \varepsilon\varepsilon_0 \langle E^2 \rangle = \frac{\varepsilon\varepsilon_0}{2} A_0^2$$



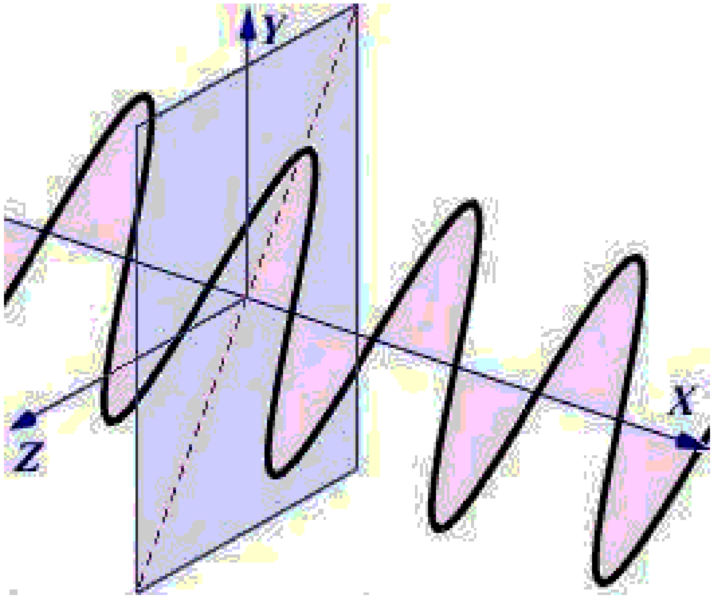
$$\nu = (\mu\mu_0\varepsilon\varepsilon_0)^{-1/2}$$

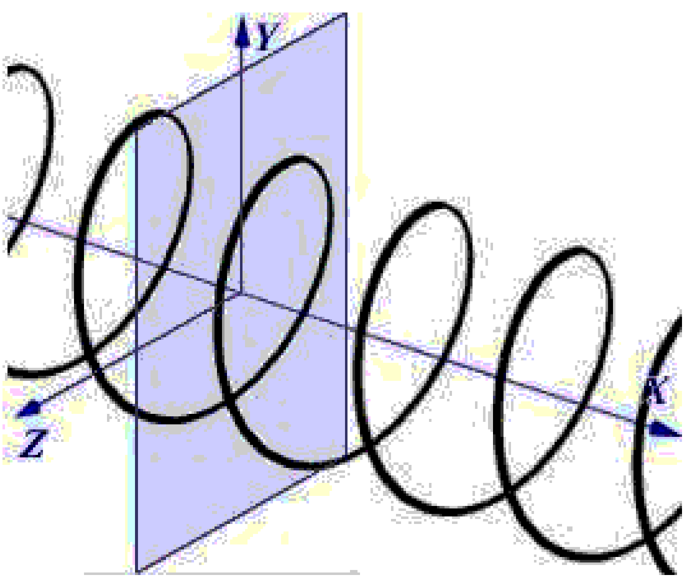
$$S\nu = w$$

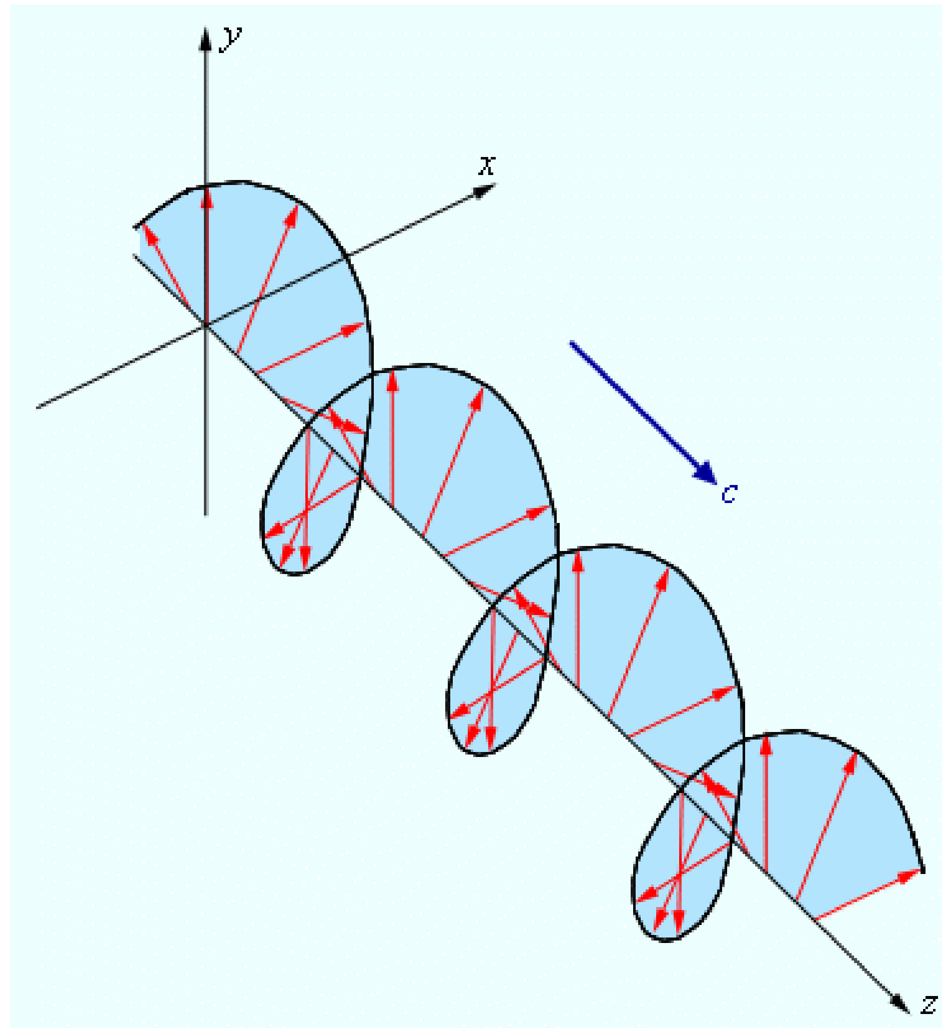


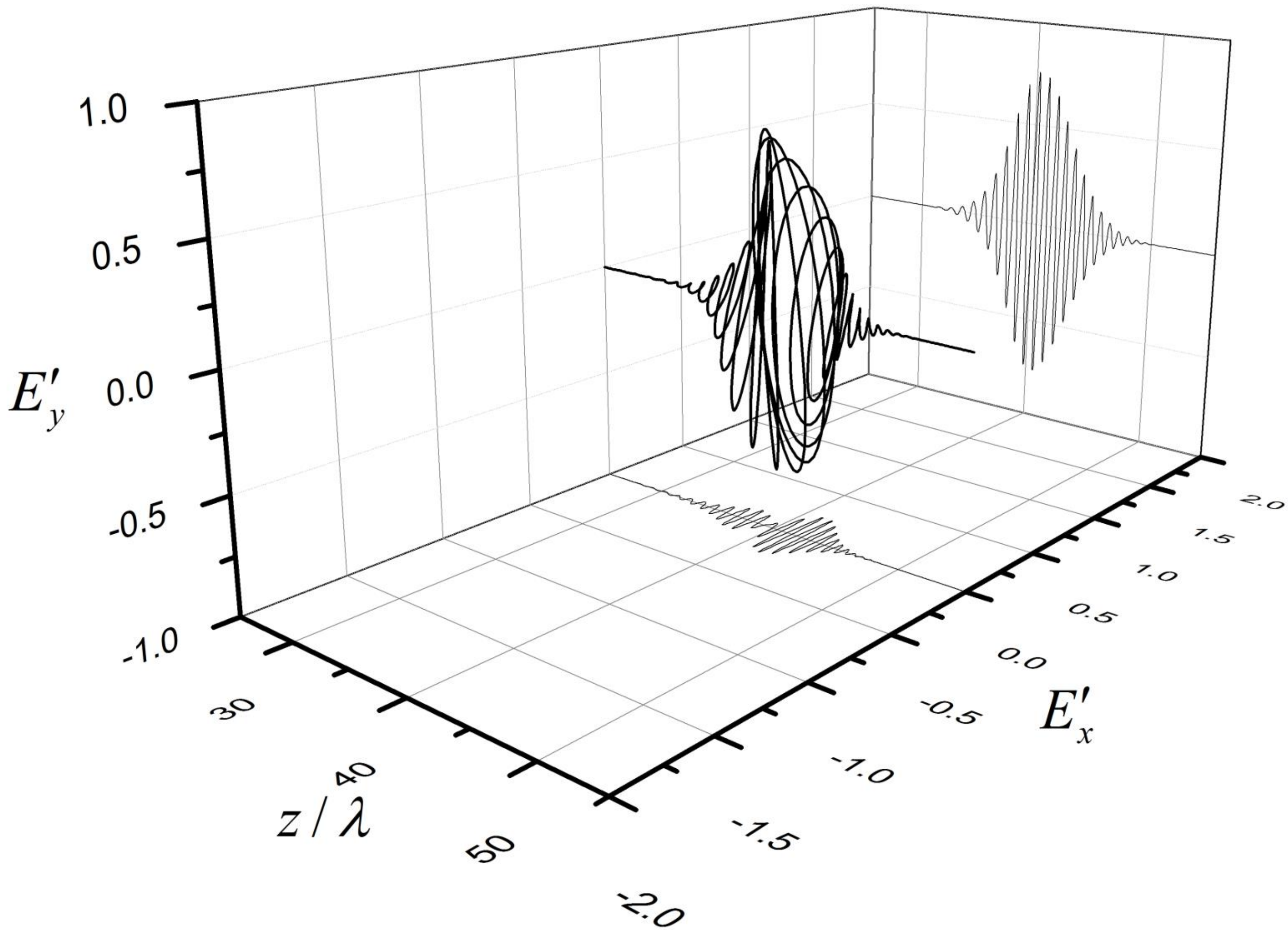


$\Delta\varphi = 0$ $\Delta\varphi = \pi / 6$ $\Delta\varphi = \pi / 2$ $\Delta\varphi = 5\pi / 6$ $\Delta\varphi = \pi$ $\Delta\varphi = 7\pi / 6$


Y
X
Z


Y
X
Z


y
x
c
z



1.0
0.5
0.0
-0.5
-1.0
E'_y
30
40
50
z / λ
2.0
1.5
1.0
0.5
0.0
-0.5
-1.0
-1.5
-2.0
E'_x

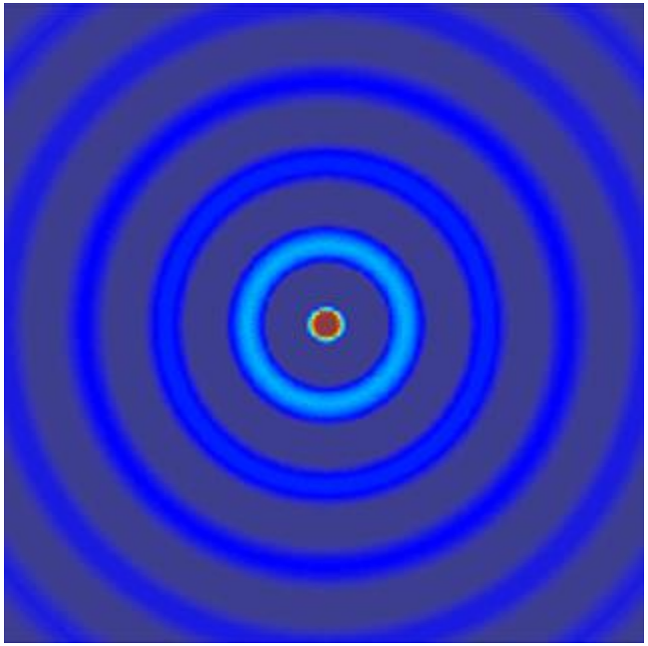
$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \mathbf{E}}{\partial r}\right)-\frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}=0$$

$$\frac{\partial^2 \mathbf{E}}{\partial r^2}+\frac{2}{r}\frac{\partial \mathbf{E}}{\partial r}-\frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}=0$$

$$\frac{1}{r}\frac{\partial^2 (r\mathbf{E})}{\partial r^2}-\frac{1}{v^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}=0$$

$$\mathbf{E}=\frac{\mathbf{F}_1(t-r/v)}{r}+\frac{\mathbf{F}_2(t+r/v)}{r}$$

$$\mathbf{E}=\frac{\mathbf{A}_1\sin[\omega(t-r/v)+\varphi_1]}{r}$$



Скалярный и векторный потенциалы переменных во времени полей

$$\operatorname{div}\mathbf{B}(\mathbf{r},t)=0 \quad\Rightarrow\quad \mathbf{B}(\mathbf{r},t)=\operatorname{rot}\mathbf{A}(\mathbf{r},t)$$

$$\operatorname{rot}\mathbf{E}(\mathbf{r},t)=-\frac{\partial}{\partial t}\mathbf{B}(\mathbf{r},t)=-\frac{\partial}{\partial t}\operatorname{rot}\mathbf{A}(\mathbf{r},t)=-\operatorname{rot}\frac{\partial\mathbf{A}}{\partial t}$$

$$\operatorname{rot}(\mathbf{E}+\partial\mathbf{A}/\partial t)=0 \quad\Rightarrow\quad \mathbf{E}+\partial\mathbf{A}/\partial t=-\operatorname{grad}\varphi(\mathbf{r},t)$$

$$\boxed{\mathbf{E}=-\partial\mathbf{A}(\mathbf{r},t)/\partial t-\operatorname{grad}\varphi(\mathbf{r},t) \qquad \mathbf{B}=\operatorname{rot}\mathbf{A}(\mathbf{r},t)}$$

$$\mathbf{A}_1=\mathbf{A}+\operatorname{grad}\vartheta(\mathbf{r},t) \qquad \varphi_1=\varphi-\partial\vartheta(\mathbf{r},t)/\partial t$$

$$\operatorname{rot}\mathbf{A}_1(\mathbf{r},t)=\mathbf{B}(\mathbf{r},t) \qquad \mathbf{E}=-\partial\mathbf{A}_1(\mathbf{r},t)/\partial t-\operatorname{grad}\varphi_1(\mathbf{r},t)$$

$$\mathbf{E} = -\partial \mathbf{A}_1(\mathbf{r},t)/\partial t - \operatorname{grad}\varphi_1(\mathbf{r},t) = -\partial \mathbf{A}(\mathbf{r},t)/\partial t -$$

$$-\cancel{\frac{\partial}{\partial t}\operatorname{grad}\vartheta} - \operatorname{grad}\varphi(\mathbf{r},t) + \cancel{\operatorname{grad}\frac{\partial\vartheta}{\partial t}} = -\partial \mathbf{A}(\mathbf{r},t)/\partial t - \operatorname{grad}\varphi(\mathbf{r},t)$$

Условие калибровки Лоренца

$$\boxed{\operatorname{div}\mathbf{A}(\mathbf{r},t) + \mu\mu_0\varepsilon\varepsilon_0\partial\varphi(\mathbf{r},t)/\partial t = 0}$$

Уравнение для скалярного и векторного потенциалов

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = \operatorname{rot}\mathbf{B} = \mu\mu_0 \operatorname{rot}\mathbf{H} = \mu\mu_0(\mathbf{j} + \varepsilon\varepsilon_0\partial\mathbf{E}/\partial t) = \mu\mu_0\mathbf{j} +$$

$$+\mu\mu_0\varepsilon\varepsilon_0\frac{\partial}{\partial t}(-\operatorname{grad}\varphi - \partial\mathbf{A}/\partial t) = \mu\mu_0\mathbf{j} - \mu\mu_0\varepsilon\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} -$$

$$-\mu\mu_0\varepsilon\varepsilon_0\operatorname{grad}(\partial\varphi/\partial t)$$

$$\operatorname{rot}\operatorname{rot}\mathbf{A} = \operatorname{grad}\operatorname{div}\mathbf{A} - \Delta\mathbf{A}$$

$$\mu\mu_0\mathbf{j} - \mu\mu_0\varepsilon\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}\overbrace{-\mu\mu_0\varepsilon\varepsilon_0\operatorname{grad}(\partial\varphi/\partial t) = \operatorname{grad}\operatorname{div}\mathbf{A}}^{=0} - \Delta\mathbf{A}$$

$$\boxed{\Delta\mathbf{A}-\mu\mu_0\varepsilon\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2}=-\mu\mu_0\mathbf{j}(\mathbf{r},t)}$$

$$\frac{\rho(\mathbf{r},t)}{\varepsilon\varepsilon_0}=\operatorname{div}\mathbf{E}=\operatorname{div}(-\operatorname{grad}\varphi-\partial\mathbf{A}/\partial t)=-\Delta\varphi-\operatorname{div}\frac{\partial}{\partial t}\mathbf{A}=$$

$$=-\Delta\varphi-\frac{\partial}{\partial t}\left(\operatorname{div}\mathbf{A}\right)=-\Delta\varphi-\frac{\partial}{\partial t}\left(-\varepsilon\varepsilon_0\mu\mu_0\frac{\partial\varphi}{\partial t}\right)=$$

$$=-\Delta\varphi+\varepsilon\varepsilon_0\mu\mu_0\frac{\partial^2\varphi}{\partial t^2}$$

Условие калибровки

$$\boxed{\Delta\varphi-\mu\mu_0\varepsilon\varepsilon_0\frac{\partial^2\varphi}{\partial t^2}=-\rho(\mathbf{r},t)/\varepsilon\varepsilon_0}$$

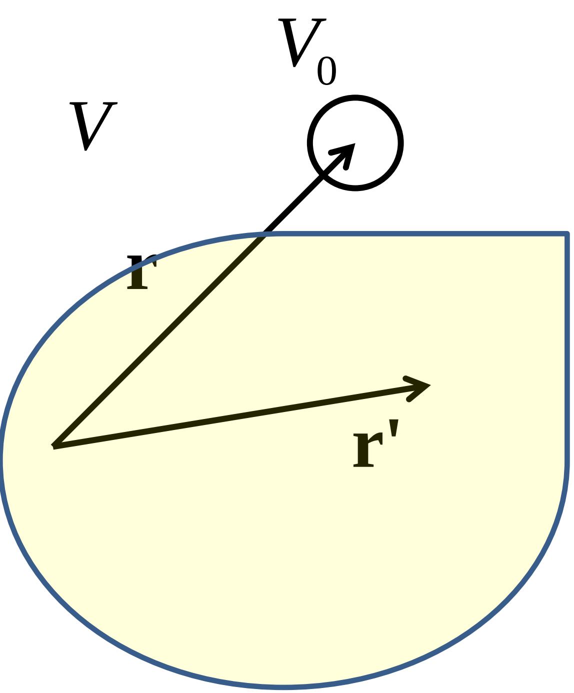
Решение уравнений для скалярного и векторного потенциалов

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu\mu_0}{4\pi}\oint\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\varphi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon\varepsilon_0}\oint\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

Надо убедиться, что выписанные выше формулы для скалярного и векторного потенциалов действительно являются решениями волновых уравнений и что они удовлетворяют условию калибровки

$$\Delta\varphi - \frac{1}{v^2}\frac{\partial^2\varphi}{\partial t^2} = -\rho(\mathbf{r},t)/\varepsilon\varepsilon_0 \Rightarrow \varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon\varepsilon_0}\oint \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$



$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon\varepsilon_0}\int_{V_0\to 0} \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' +$$

$$+\frac{1}{4\pi\varepsilon\varepsilon_0}\int_{V} \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \varphi_1 + \varphi_2$$

Оператор Даламбера $\quad \Box = \Delta - \dfrac{1}{v^2}\dfrac{\partial^2}{\partial t^2}$

$\Box\varphi_2 = 0$ Действительно $\mathbf{r}-\mathbf{r}'$ ограничено снизу

$$\Rightarrow \frac{1}{4\pi\varepsilon\varepsilon_0}\Box\int_{V} \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}' = \frac{1}{4\pi\varepsilon\varepsilon_0}\int_{V} \underbrace{\Box\left(\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}\right)}_{=0} d\mathbf{r}' = 0$$

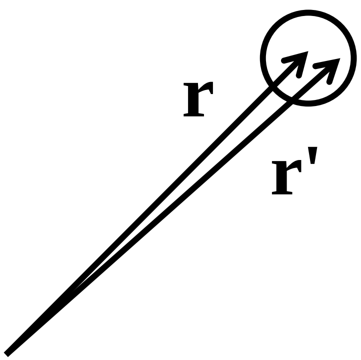
$$\frac{1}{v^2}\frac{\partial^2\varphi_1}{\partial t^2}=\frac{1}{4\pi\varepsilon\varepsilon_0}\frac{1}{v^2}\frac{\partial^2}{\partial t^2}\int_{V_0\to 0}\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'=$$

$$=\frac{1}{4\pi\varepsilon\varepsilon_0}\frac{1}{v^2}\int_{V_0\to 0}\frac{\partial^2}{\partial t^2}\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'=$$

$$=\frac{1}{4\pi\varepsilon\varepsilon_0 v^2}\frac{\partial^2}{\partial t^2}\rho(\tilde{\mathbf{r}},t-\tilde{R}/v)\int_{V_0\to 0}\frac{4\pi R^2}{R}dR$$

$$=0$$

$$\tilde{\mathbf{R}}=\tilde{\mathbf{r}}-\mathbf{r}' \qquad \mathbf{R}=\mathbf{r}-\mathbf{r}'$$



$$\Delta\varphi_1=\operatorname{div}\operatorname{grad}\varphi_1=\lim_{V_0\to 0}\frac{1}{V_0}\oint_{S_0}\operatorname{grad}(\varphi_1)d\mathbf{S}_0$$

$$\operatorname{grad}\varphi_1=\operatorname{grad}\left(\frac{1}{4\pi\varepsilon\varepsilon_0}\int_{V_0\to 0}\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right)$$

$$\operatorname{grad}\varphi_1 = \operatorname{grad}\left(\frac{1}{4\pi\varepsilon\varepsilon_0}\int\limits_{V_0\to 0}\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\right)=$$

$$=\frac{1}{4\pi\varepsilon\varepsilon_0}\int\limits_{V_0\to 0}\operatorname{grad}\left(\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}\right)d\mathbf{r}'=$$

$$=\underbrace{\cancel{\frac{1}{4\pi\varepsilon\varepsilon_0}\int\limits_{V_0\to 0}\frac{\operatorname{grad}\{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)\}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'}}_{=0}-\frac{1}{4\pi\varepsilon\varepsilon_0}\int\limits_{V_0\to 0}\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{r}'$$

т.к. $$\int\limits_{V_0\to 0}\frac{\operatorname{grad}\{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)\}}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'=\operatorname{grad}\rho(\tilde{\mathbf{r}},t-\tilde{R}/v)\int\limits_{V_0\to 0}\frac{4\pi R^2dR}{R}\to 0$$

$$\mathbf{R}=\mathbf{r}-\mathbf{r}'$$

$$\tilde{\mathbf{R}}=\tilde{\mathbf{r}}-\mathbf{r}'$$

$$\Rightarrow\quad \operatorname{divgrad}\varphi_1=\lim_{V_0\to 0}\frac{1}{V_0}\oint\limits_{S_0}\operatorname{grad}(\varphi_1)d\mathbf{S}_0=-\frac{1}{4\pi\varepsilon\varepsilon_0}\times$$

$$\times\lim_{V_0\to 0}\frac{1}{V_0}\oint\limits_{S_0}\int\limits_{V_0\to 0}\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}d\mathbf{S}_0d\mathbf{r}'=$$

$$=-\frac{1}{4\pi\varepsilon\varepsilon_0}\times\lim_{V_0\to 0}\frac{1}{V_0}\int\limits_{V_0\to 0}\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)d\mathbf{r}'\underbrace{\int\limits_{S_0}\frac{(\mathbf{r}-\mathbf{r}')d\mathbf{S}_0}{|\mathbf{r}-\mathbf{r}'|^3}}_{=4\pi}$$

Меняем порядок интегрирования

$$\Delta\varphi_1 = -\frac{1}{\varepsilon\varepsilon_0}\times\lim_{V_0\to 0}\frac{1}{V_0}\int_{V_0\to 0}\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/v)d\mathbf{r}' =$$

$$= -\frac{1}{\varepsilon\varepsilon_0}\times\lim_{V_0\to 0}\rho(\tilde{\mathbf{r}}, t-\tilde{R}/v)\frac{1}{V_0}\int_{V_0\to 0}d\mathbf{r}' =$$

$$\mathbf{R} = \mathbf{r}-\mathbf{r}'$$

$$\tilde{\mathbf{R}} = \tilde{\mathbf{r}}-\mathbf{r}'$$

$$= -\frac{1}{\varepsilon\varepsilon_0}\times\lim_{V_0\to 0}\rho(\tilde{\mathbf{r}}, t-\tilde{R}/v) =$$

$$= -\frac{1}{\varepsilon\varepsilon_0}\times\lim_{V_0\to 0}\rho(\tilde{\mathbf{r}}, t-|\tilde{\mathbf{r}}-\mathbf{r}'|/v) = -\frac{1}{\varepsilon\varepsilon_0}\rho(\mathbf{r}, t)$$

$$V_0 \to 0 \quad \leftrightarrow \quad \tilde{\mathbf{r}}\to\mathbf{r}, \quad \mathbf{r}'\to\mathbf{r}$$

$$\Delta\varphi - \mu\mu_0\varepsilon\varepsilon_0\frac{\partial^2\varphi}{\partial t^2} = -\rho(\mathbf{r}, t)/\varepsilon\varepsilon_0 \qquad \varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon\varepsilon_0}\oint\frac{\rho(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\Delta\mathbf{A} - \mu\mu_0\varepsilon\varepsilon_0\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mu_0\mathbf{j}(\mathbf{r}, t) \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu\mu_0}{4\pi}\oint\frac{\mathbf{j}(\mathbf{r}', t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\operatorname{div}\mathbf{A}(\mathbf{r},t)+\mu\mu_0\varepsilon\varepsilon_0\partial\varphi(\mathbf{r},t)/\partial t=0\ ?$$

$$\operatorname{div}\mathbf{A}(\mathbf{r},t)=\frac{\mu\mu_0}{4\pi}\oint\operatorname{div}\left(\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}\right)d\mathbf{r}'\ \Rightarrow$$

$$\operatorname{div}\psi\mathbf{a}=\psi\operatorname{div}\mathbf{a}+\mathbf{a}\cdot\operatorname{grad}\psi\qquad \operatorname{div}\mathbf{F}(f(\mathbf{r}))=\frac{\partial\mathbf{F}}{\partial f}\operatorname{grad}f(\mathbf{r})$$

$$t'(\mathbf{r})=t-|\mathbf{r}-\mathbf{r}'|/v$$

$$\operatorname{div}\left(\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}\mathbf{j}(\mathbf{r}',t')+\mathbf{j}(\mathbf{r}',t')\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}=$$

$$=\underline{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\mathbf{j}(\mathbf{r}',t')}{\partial t'}\left(-\frac{1}{v}\right)\operatorname{grad}|\mathbf{r}-\mathbf{r}'|}+\underline{\mathbf{j}(\mathbf{r}',t')\operatorname{grad}\frac{1}{|\mathbf{r}-\mathbf{r}'|}}$$

$$\operatorname{div}'\left(\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\operatorname{div}'\mathbf{j}(\mathbf{r}',t')_{t'=\mathrm{const}}+$$

Сложем две дивергенции. Подчеркнутые одним цветом слагаемые отличаются знаком

$$+\underline{\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\mathbf{j}(\mathbf{r}',t')}{\partial t'}\left(-\frac{1}{v}\right)\operatorname{grad}'|\mathbf{r}-\mathbf{r}'|}+\underline{\mathbf{j}(\mathbf{r}',t')\cdot\operatorname{grad}'\frac{1}{|\mathbf{r}-\mathbf{r}'|}}$$

Сложим две дивергенции

$$\mathrm{div}\left(\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right)+\mathrm{div}'\left(\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right)=\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{div}'\,\mathbf{j}(\mathbf{r}',t')_{t'=\mathrm{const}}$$

$$\mathrm{div}\,\mathbf{A}(\mathbf{r},t)=\frac{\mu\mu_0}{4\pi}\oint \mathrm{div}\left(\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right)d\mathbf{r}'=-\frac{\mu\mu_0}{4\pi}\oint \mathrm{div}'\left(\frac{\mathbf{j}(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}\right)d\mathbf{r}'+$$

$$+\frac{\mu\mu_0}{4\pi}\oint\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{div}'\,\mathbf{j}(\mathbf{r}',t')_{t'=\mathrm{const}}\,d\mathbf{r}'= \qquad t'=t-|\mathbf{r}-\mathbf{r}'|/v$$

$$=\underbrace{-\frac{\mu\mu_0}{4\pi}\oint_S\frac{j_n(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}dS}_{=0}+\frac{\mu\mu_0}{4\pi}\oint\frac{1}{|\mathbf{r}-\mathbf{r}'|}\mathrm{div}'\,\mathbf{j}(\mathbf{r}',t')_{t'=\mathrm{const}}\,d\mathbf{r}'$$

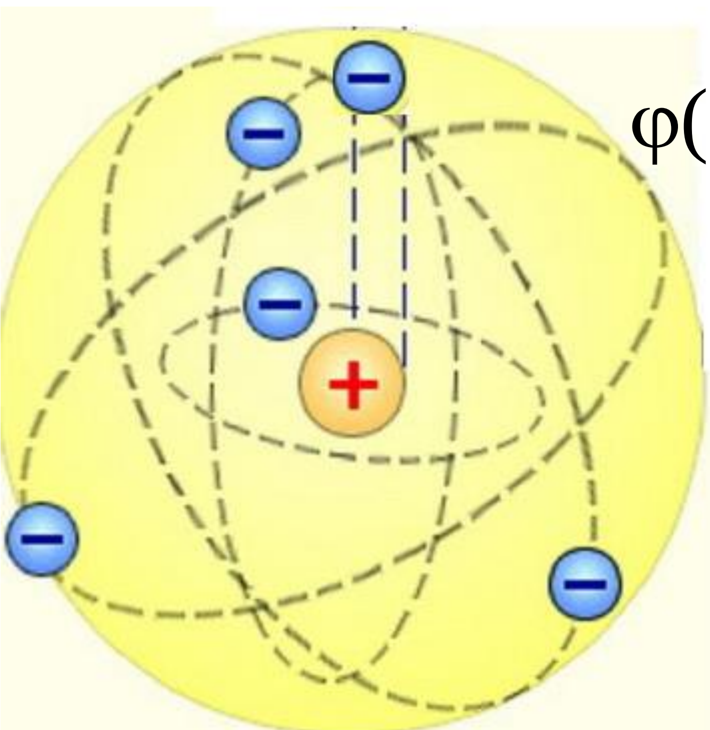
Вычислим теперь

$$\mu\mu_0\varepsilon\varepsilon_0\frac{\partial\varphi(\mathbf{r},t)}{\partial t}=\frac{\mu\mu_0}{4\pi}\frac{\partial}{\partial t}\oint\frac{\rho(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'=\frac{\mu\mu_0}{4\pi}\oint\frac{1}{|\mathbf{r}-\mathbf{r}'|}\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}d\mathbf{r}'$$

Сложим подчеркнутые красной чертой слагаемые

$$\operatorname{div}\mathbf{A}(\mathbf{r},t)+\mu\mu_0\varepsilon\varepsilon_0\frac{\partial\varphi(\mathbf{r},t)}{\partial t}=$$

$$=\frac{\mu\mu_0}{4\pi}\oint\underbrace{\left\{\frac{\partial\rho(\mathbf{r}',t')}{\partial t'}+\operatorname{div}'\mathbf{j}(\mathbf{r}',t')\right\}}_{=0}\frac{d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}=0$$



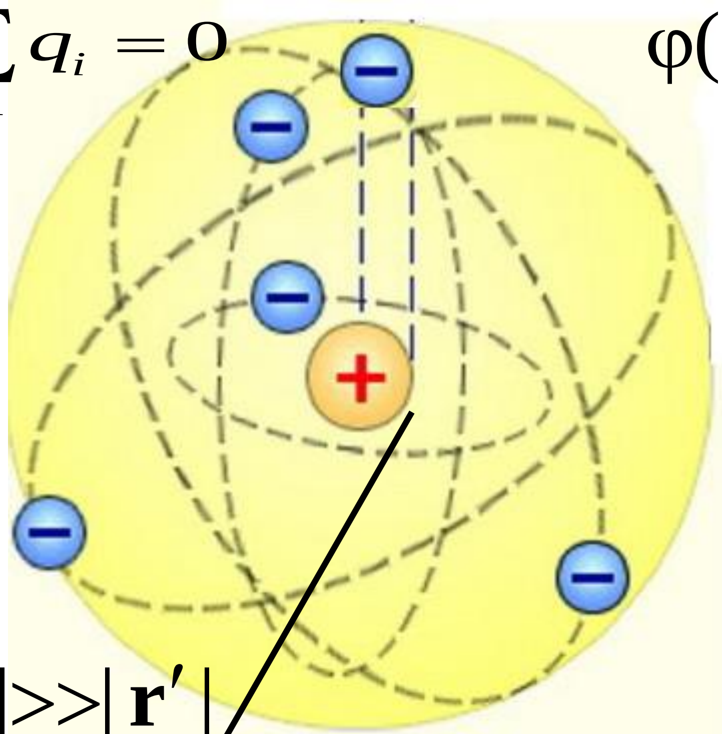
$$\varphi(\mathbf{r},t)=\frac{1}{4\pi\varepsilon\varepsilon_0}\oint\frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu\mu_0}{4\pi}\oint\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$\mathbf{E}=-\partial\mathbf{A}(\mathbf{r},t)/\partial t-\operatorname{grad}\varphi(\mathbf{r},t)$$

$$\mathbf{B}(\mathbf{r},t)=\operatorname{rot}\mathbf{A}(\mathbf{r},t)$$

$$\sum_{i=1}^{n} q_i = 0$$



$$|\mathbf{r}| >> |\mathbf{r}'|$$

$\mathbf{r}$

$$\varphi(\mathbf{r},t) = \frac{1}{4\pi\varepsilon\varepsilon_0}\oint \frac{\rho(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'$$

$$|\mathbf{r}-\mathbf{r}'| \approx r - \frac{\mathbf{r}\mathbf{r}'}{r}$$

$= 0$

$$\varphi(\mathbf{r},t) \approx \frac{1}{4\pi\varepsilon\varepsilon_0 r}\oint \rho(\mathbf{r}',t-r/v)d\mathbf{r}' -$$

$$-\frac{1}{4\pi\varepsilon\varepsilon_0}\oint \frac{\mathbf{r}\cdot\mathbf{r}'}{r}\frac{\partial}{\partial r}\left(\frac{\rho(\mathbf{r}',t-r/v)}{r}\right)d\mathbf{r}'$$

$$\varphi(\mathbf{r},t) \approx -\frac{1}{4\pi\varepsilon\varepsilon_0}\frac{\mathbf{r}}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\oint \mathbf{r}'\rho(\mathbf{r}',t-r/v)d\mathbf{r}'\right) =$$

$$= -\frac{1}{4\pi\varepsilon\varepsilon_0}\frac{\mathbf{r}}{r}\frac{\partial}{\partial r}\left(\frac{\mathbf{p}(t-r/v)}{r}\right) = -\frac{1}{4\pi\varepsilon\varepsilon_0}\left(-\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \frac{\mathbf{r}}{r^2}\frac{\partial \mathbf{p}}{\partial r}\right) =$$

$$= -\frac{1}{4\pi\varepsilon\varepsilon_0}\left(\mathbf{p}\cdot\mathrm{grad}\left(\frac{1}{r}\right) + \frac{1}{r}\mathrm{div}\,\mathbf{p}\right) \Rightarrow$$

?

$$\frac{\mathbf{r}}{r}\frac{\partial \mathbf{p}}{\partial r}=\frac{x}{r}\frac{\partial p_x}{\partial r}+\frac{y}{r}\frac{\partial p_y}{\partial r}+\frac{z}{r}\frac{\partial p_z}{\partial r}=\frac{\partial r}{\partial x}\frac{\partial p_x}{\partial r}+\frac{\partial r}{\partial y}\frac{\partial p_y}{\partial r}+\frac{\partial r}{\partial z}\frac{\partial p_z}{\partial r}=\operatorname{div}\mathbf{p}$$

$$\varphi(\mathbf{r},t)\approx-\frac{1}{4\pi\varepsilon\varepsilon_0}\left(\mathbf{p}\cdot\operatorname{grad}\left(\frac{1}{r}\right)+\frac{1}{r}\operatorname{div}\mathbf{p}\right)=-\frac{1}{4\pi\varepsilon\varepsilon_0}\operatorname{div}\left(\frac{\mathbf{p}(t-r/v)}{r}\right)$$

$$\operatorname{div}\alpha\mathbf{a}=\mathbf{a}\cdot\operatorname{grad}\alpha+\alpha\operatorname{div}\mathbf{a}$$

$$\mathbf{A}(\mathbf{r},t)=\frac{\mu\mu_0}{4\pi}\oint\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/v)}{|\mathbf{r}-\mathbf{r}'|}d\mathbf{r}'\approx\frac{\mu\mu_0}{4\pi}\oint\frac{\mathbf{j}(\mathbf{r}',t-r/v)}{r}d\mathbf{r}'=$$

$$=\frac{\mu\mu_0}{4\pi r}\oint\mathbf{j}(\mathbf{r}',t-r/v)d\mathbf{r}'\Rightarrow \textit{приводится к}\Rightarrow\frac{\mu\mu_0}{4\pi r}\frac{\partial\mathbf{p}(t-r/v)}{\partial t}$$

Уравнение непрерывности $\Rightarrow$ $$\frac{\partial\rho(\mathbf{r}',t-r/v)}{\partial t}=-\operatorname{div}'\mathbf{j}(\mathbf{r}',t-r/v)$$

Умножим на **r’** и проинтегрируем

$$\int_V\mathbf{r}'\frac{\partial\rho(\mathbf{r}',t-r/v)}{\partial t}d\mathbf{r}'=\underline{\frac{\partial}{\partial t}\mathbf{p}(t-r/v)}=\underline{-\int_V\mathbf{r}'\operatorname{div}'\mathbf{j}(\mathbf{r}',t-r/v)d\mathbf{r}'}$$

Умножим подчеркнутые члены на постоянный вектор **a** и проинтегрируем

$$\mathbf{a}\frac{\partial}{\partial t}\mathbf{p} = -\mathbf{a}\int\limits_V \mathbf{r}'\,\mathrm{div}'\,\mathbf{j}(\mathbf{r}', t - r/v)\,d\mathbf{r}' = -\int\limits_V \mathbf{a}\mathbf{r}'\,\mathrm{div}'\,\mathbf{j}(\mathbf{r}', t - r/v)\,d\mathbf{r}' =$$

$$= -\int\limits_V \mathrm{div}'\big((\mathbf{a}\cdot\mathbf{r}')\mathbf{j}\big)\,d\mathbf{r}' + \int\limits_V \mathbf{j}\,\mathrm{grad}'\big(\mathbf{a}\cdot\mathbf{r}'\big)\,d\mathbf{r}' =$$

$$\mathrm{grad}'(\mathbf{a}\cdot\mathbf{r}') = \mathbf{a}$$

$$= -\underbrace{\oint\limits_S \big(\mathbf{a}\cdot\mathbf{r}'\big)\,j_n\,d\mathbf{r}'}_{=0} + \mathbf{a}\int\limits_V \mathbf{j}(\mathbf{r}', t - r/v)\,d\mathbf{r}'$$

$$\mathrm{div}\,\alpha\mathbf{a} = \mathbf{a}\cdot\mathrm{grad}\,\alpha + \alpha\,\mathrm{div}\,\mathbf{a}$$

$$\boxed{\varphi(\mathbf{r},t) \approx -\frac{1}{4\pi\varepsilon\varepsilon_0}\mathrm{div}\left(\frac{\mathbf{p}(t - r/v)}{r}\right) \qquad \mathbf{A}(\mathbf{r},t) \approx \frac{\mu\mu_0}{4\pi r}\frac{\partial\mathbf{p}(t - r/v)}{\partial t}}$$

Вычислим **B** и **E**

$$\frac{\mathbf{p}(t - r/v)}{r} = \mathbf{p}_0 f(\mathbf{r},t)$$

$$\varphi(\mathbf{r},t) \approx -\frac{1}{4\pi\varepsilon\varepsilon_0}\mathrm{div}\,\mathbf{p}_0 f(\mathbf{r},t) \qquad \mathbf{A}(\mathbf{r},t) \approx \frac{\mu\mu_0}{4\pi}\frac{\partial}{\partial t}\mathbf{p}_0 f(\mathbf{r},t)$$

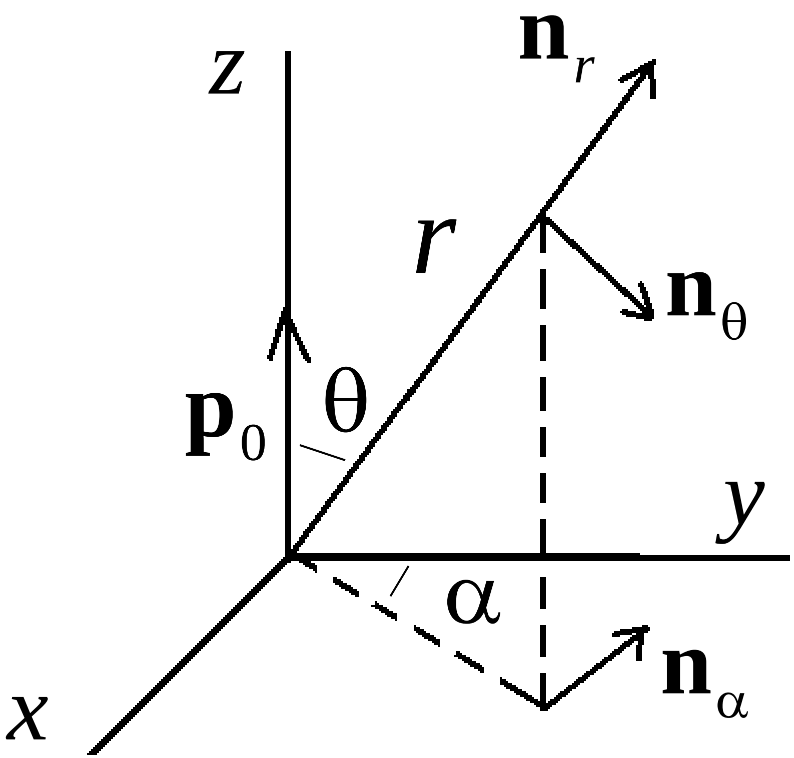
$$\mathbf{B} = \mathrm{rot}\,\mathbf{A} = \frac{\mu\mu_0}{4\pi}\mathrm{rot}\frac{\partial}{\partial t}\mathbf{p}_0 f(\mathbf{r},t) = \frac{\mu\mu_0}{4\pi}\frac{\partial}{\partial t}\mathrm{rot}\,\mathbf{p}_0 f(\mathbf{r},t)$$

$$\mathbf{E} = -\operatorname{grad}\varphi - \partial\mathbf{A}/\partial t = \frac{1}{4\pi\varepsilon\varepsilon_0}\operatorname{grad}\operatorname{div}\mathbf{p}_0 f(r,t) - \frac{\partial}{\partial t}\frac{\mu\mu_0}{4\pi}\frac{\partial}{\partial t}\mathbf{p}_0 f(r,t) =$$

$$= \frac{1}{4\pi\varepsilon\varepsilon_0}\left(\operatorname{rot}\operatorname{rot}\mathbf{p}_0 f(r,t) + \underbrace{\Delta\mathbf{p}_0 f(r,t) - \frac{1}{v^2}\frac{\partial^2}{\partial t^2}\mathbf{p}_0 f(r,t)}_{=0}\right) =$$

$$= \frac{1}{4\pi\varepsilon\varepsilon_0}\operatorname{rot}\operatorname{rot}\mathbf{p}_0 f(r,t)$$

$$\frac{\mathbf{p}(t-r/v)}{r} = \mathbf{p}_0 f(\mathbf{r},t)$$



$$p_{0r} = p_0\cos\theta$$

$$p_{0\theta} = -p_0\sin\theta$$

$$p_{0\alpha} = 0$$

$$\frac{\mathbf{p}(t-r/v)}{r} = \mathbf{p}_0 f(\mathbf{r},t) = \mathbf{p}_0\frac{\exp[i\omega(t-r/v)]}{r}$$

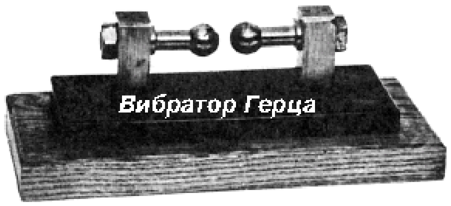
$$\operatorname{rot}\mathbf{p}_0 f = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta\cdot p_{0\alpha} f) - \frac{\partial}{\partial\alpha}(p_{0\theta} f)\right]\mathbf{n}_r +$$

$$+\frac{1}{r}\left[\sin\theta\frac{\partial}{\partial\alpha}(p_{0r} f) - \frac{\partial}{\partial r}(rp_{0\alpha} f)\right]\mathbf{n}_\theta +$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}(rp_{0\theta} f) - \frac{\partial}{\partial\theta}(p_{0r} f)\right]\mathbf{n}_\alpha$$

$$p_{0r} = p_0\cos\theta$$
$$p_{0\theta} = -p_0\sin\theta$$
$$p_{0\alpha} = 0$$

$$f(\mathbf{r},t) = \frac{\exp[i\omega(t - r/v)]}{r} \qquad \frac{\partial}{\partial r} f(\mathbf{r},t) = -\frac{1}{r} f - \frac{i\omega}{v} f$$



$$\left(\operatorname{rot}\mathbf{p}_0 f\right)_r = 0 \qquad \left(\operatorname{rot}\mathbf{p}_0 f\right)_\theta = 0$$

$$\left(\operatorname{rot}\mathbf{p}_0 f\right)_\alpha = \frac{1}{r}\left[-\frac{\partial}{\partial r}(rp_0\sin\theta\cdot f) + p_0\sin\theta\cdot f\right] = -p_0\sin\theta\frac{\partial f}{\partial r} =$$

$$= \frac{p_0\sin\theta}{r}\left(\frac{1}{r} + \frac{i\omega}{v}\right)\exp\left(i\omega(t - r/v)\right)$$

$$\operatorname{rot}\operatorname{rot}\mathbf{p}_0 f = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\underline{(\operatorname{rot}\mathbf{p}_0 f)_\alpha}\right) - \frac{\partial}{\partial\alpha}\cancel{(\operatorname{rot}\mathbf{p}_0 f)_\theta}\right]\mathbf{n}_r +$$

$$+\frac{1}{r}\left[\sin\theta\frac{\partial}{\partial\alpha}\cancel{(\operatorname{rot}\mathbf{p}_0 f)_r} - \frac{\partial}{\partial r}\left(r\underline{(\operatorname{rot}\mathbf{p}_0 f)_\alpha}\right)\right]\mathbf{n}_\theta +$$

$$+\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\cancel{(\operatorname{rot}\mathbf{p}_0 f)_\theta}\right) - \frac{\partial}{\partial\theta}\cancel{(\operatorname{rot}\mathbf{p}_0 f)_r}\right]\mathbf{n}_\alpha$$

$$(\operatorname{rot}\mathbf{p}_0 f)_r = 0$$

$$(\operatorname{rot}\mathbf{p}_0 f)_\theta = 0$$

$$\mathbf{E} = \frac{1}{4\pi\varepsilon\varepsilon_0}\operatorname{rot}\operatorname{rot}\mathbf{p}_0 f(r,t) \Rightarrow E_\alpha = 0$$

$$E_r = \frac{1}{4\pi\varepsilon\varepsilon_0 r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\underbrace{\frac{p_0\sin\theta}{r}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\exp\left(i\omega(t-r/v)\right)}_{(\operatorname{rot}\mathbf{p}_0 f)_\alpha}\right) =$$

$$= \frac{p_0\cos\theta}{2\pi\varepsilon\varepsilon_0 r^2}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\exp\left(i\omega(t-r/v)\right)$$

$$E_\theta = -\frac{1}{4\pi\varepsilon\varepsilon_0 r}\frac{\partial}{\partial r}\left(r\cdot \underline{\frac{p_0\sin\theta}{r}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\exp\left(i\omega(t-r/v)\right)}\right)=$$

$$= -\frac{p_0\sin\theta}{4\pi\varepsilon\varepsilon_0 r}\left(-\frac{1}{r^2}-\frac{i\omega}{v}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\right)\exp\left(i\omega(t-r/v)\right)=$$

$$= \frac{p_0\sin\theta}{4\pi\varepsilon\varepsilon_0 r}\left(\frac{1}{r^2}+\frac{i\omega}{rv}-\frac{\omega^2}{v^2}\right)\exp\left(i\omega(t-r/v)\right)$$

$$\left(\operatorname{rot}\mathbf{p}_0 f\right)_r = 0$$

$$\left(\operatorname{rot}\mathbf{p}_0 f\right)_\theta = 0$$

$$\mathbf{B} = \frac{\mu\mu_0}{4\pi}\frac{\partial}{\partial t}\operatorname{rot}\mathbf{p}_0 f(\mathbf{r},t)$$

$$\left(\operatorname{rot}\mathbf{p}_0 f\right)_\alpha = \frac{p_0\sin\theta}{r}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\exp\left(i\omega(t-r/v)\right)$$

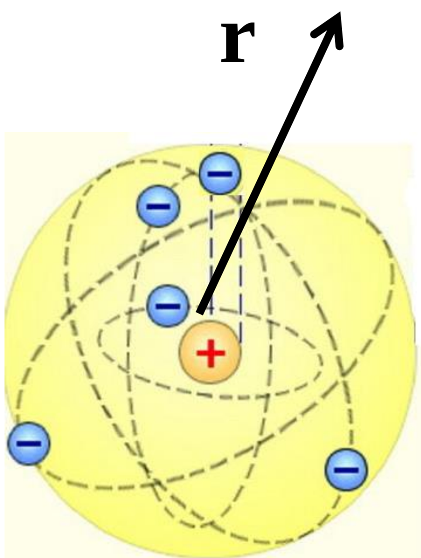
$$B_\alpha = \frac{\mu\mu_0}{4\pi}\frac{\partial}{\partial t}\left(\frac{p_0\sin\theta}{r}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\exp\left(i\omega(t-r/v)\right)\right)=$$

$$B_r = 0$$

$$= \frac{i\mu\mu_0\omega p_0\sin\theta}{4\pi rv}\left(\frac{1}{r}+\frac{i\omega}{v}\right)\exp\left(i\omega(t-r/v)\right)$$

$$B_\theta = 0$$

$$\mathbf{B}=\{0,0,B_\alpha\} \qquad \mathbf{E}=\{E_r,E_\theta,0\} \qquad \Longrightarrow \qquad \mathbf{B}\perp\mathbf{E}$$



$$|\mathbf{r}|>>|\mathbf{r}'|$$

$$\boxed{\lambda << r \iff \frac{1}{r} << \frac{\omega}{v}}$$

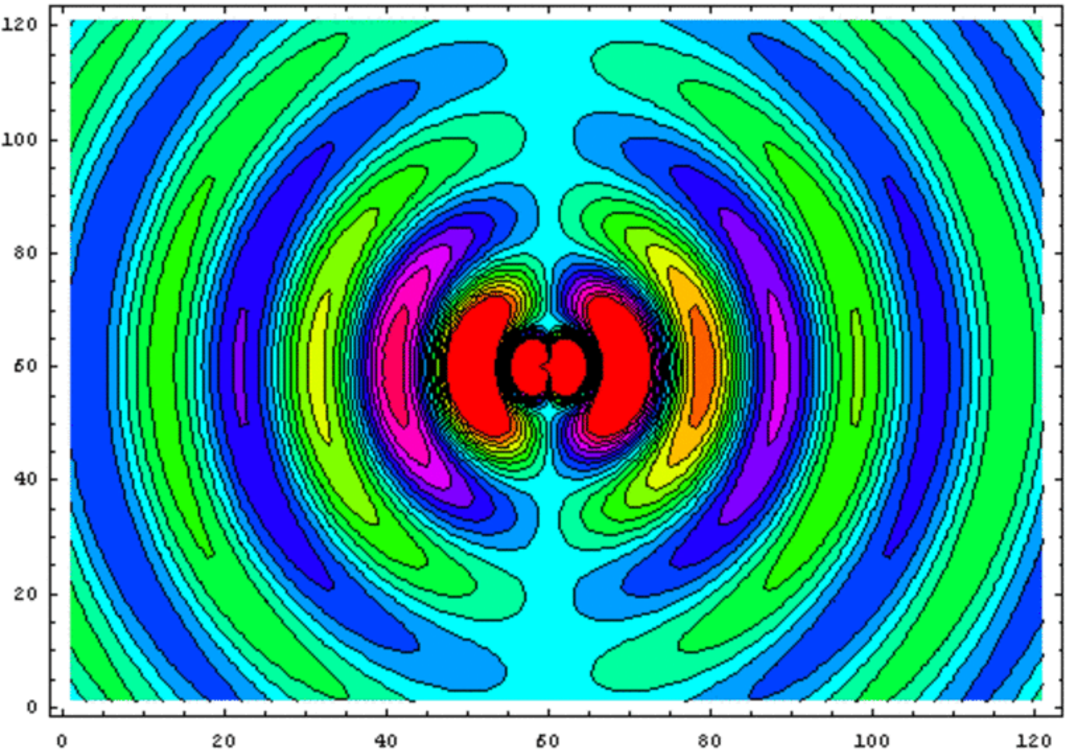
$$H_\alpha = -\frac{\omega^2 p_0 \sin\theta}{4\pi r v}\exp\left(i\omega(t-r/v)\right)$$

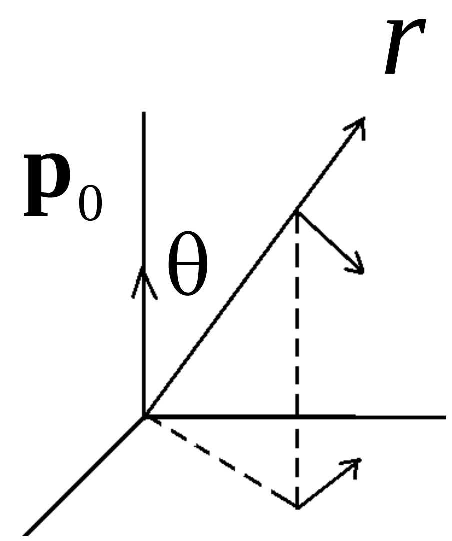
$$E_\theta = -\frac{p_0\omega^2 \sin\theta}{4\pi\varepsilon\varepsilon_0 r v^2}\exp\left(i\omega(t-r/v)\right)$$

$$\mathbf{S}=[\mathbf{E}\times\mathbf{H}]$$

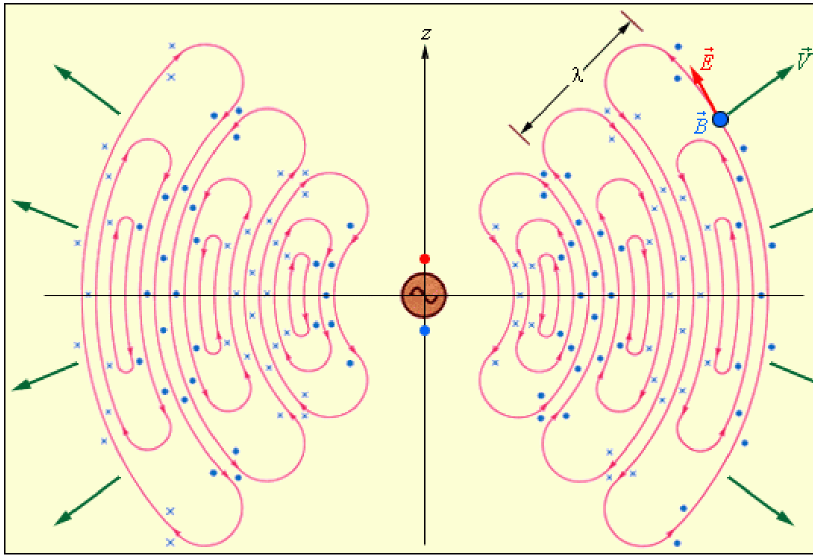
$$S_r = \mathrm{Re}\{E_\theta\}\mathrm{Re}\{H_\alpha\} = \frac{\omega^4 p_0^2 \sin^2\theta}{16\pi^2\varepsilon\varepsilon_0 r^2 v^3}\cos^2\left(\omega(t-r/v)\right)$$

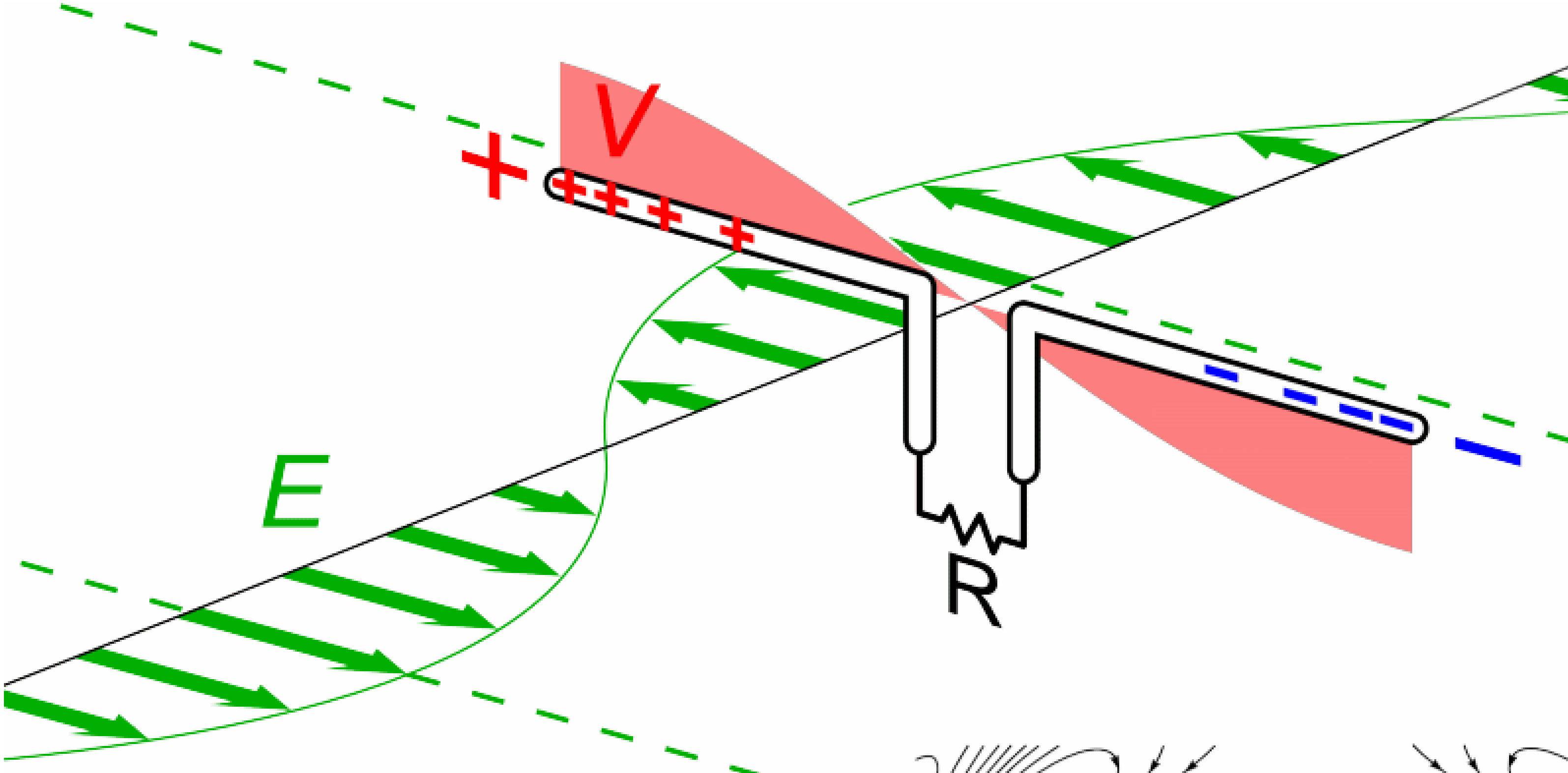
$$<S_r>=\frac{\omega^4 p_0^2 \sin^2\theta}{16\pi^2\varepsilon\varepsilon_0 r^2 v^3}<\cos^2\left(\omega(t-r/v)\right)>=\boxed{\frac{\omega^4 p_0^2 \sin^2\theta}{32\pi^2\varepsilon\varepsilon_0 r^2 v^3}}$$





$$\langle S_r \rangle = \frac{\omega^4 p_0^2 \sin^2\theta}{32\pi^2 \varepsilon\varepsilon_0 r^2 v^3}$$




V
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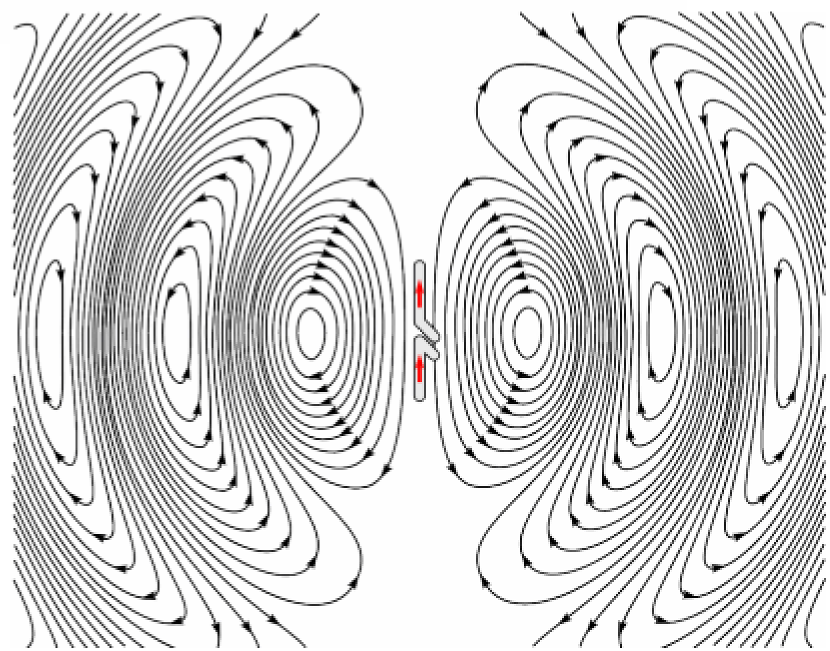


$$< S_r >= \frac{\omega^4 p_0^2 \sin^2\theta}{16\pi^2\varepsilon\varepsilon_0 r^2 v^3} < \cos^2\left(\omega(t - r/v)\right) >= \boxed{\frac{\omega^4 p_0^2 \sin^2\theta}{32\pi^2\varepsilon\varepsilon_0 r^2 v^3}}$$

Поток электромагнитной энергии сквозь поверхность сферы радиуса r

$$P = \int_0^{\pi} \sin\theta d\theta \int_0^{2\pi} d\alpha r^2 < S_r > = \frac{\omega^4 p_0^2}{32\pi^2\varepsilon\varepsilon_0 v^3} \int_0^{2\pi} d\alpha \int_0^{\pi} \sin^3\theta d\theta =$$

$$\cos\theta = x$$

$$\boxed{= \frac{\omega^4 p_0^2}{12\pi\varepsilon\varepsilon_0 v^3}}$$

$$\int_0^{\pi} \sin^3\theta d\theta = \int_1^{-1} (x^2 - 1) dx = -\frac{2}{3} - (-2) = \frac{4}{3}$$

Закон сохранения импульса

$$\frac{d\mathbf{p}_0}{dt} = q_0\{\mathbf{E} + [\mathbf{v}\times\mathbf{B}]\} \qquad n\frac{d\mathbf{p}_0}{dt} = nq_0\{\mathbf{E} + [\mathbf{v}\times\mathbf{B}]\}$$

$$\frac{d\mathbf{p}}{dt} = \oint_V \rho(\mathbf{r},t)\{\mathbf{E} + [\mathbf{v}\times\mathbf{B}]\}d\mathbf{r}$$

$$\rho = \operatorname{div}\mathbf{D} \qquad \mathbf{j} = \rho\mathbf{v} = \operatorname{rot}\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$$

$$\frac{d\mathbf{p}}{dt} = \oint_V \mathbf{E}\operatorname{div}\mathbf{D}dv + \oint_V [\operatorname{rot}\mathbf{H}\times\mathbf{B}]dv - \oint_V [\partial\mathbf{D}/\partial t\times\mathbf{B}]dv +$$

$$+\oint_V [(\operatorname{rot}\mathbf{E} + \partial\mathbf{B}/\partial t)\times\mathbf{D}]dv + \oint_V \mathbf{H}\operatorname{div}\mathbf{B}dv \Rightarrow$$

$$= 0 \qquad\qquad = 0$$

$$\frac{d\mathbf{p}}{dt} = \oint_V \mathbf{E}\,\mathrm{div}\mathbf{D}\,dv + \oint_V [\mathrm{rot}\,\mathbf{H} \times \mathbf{B}]\,dv - \oint_V [\partial\mathbf{D}/\partial t \times \mathbf{B}]\,dv +$$

$$+ \oint_V [(\mathrm{rot}\,\mathbf{E} + \partial\mathbf{B}/\partial t) \times \mathbf{D}]\,dv + \oint_V \mathbf{H}\,\mathrm{div}\mathbf{B}\,dv$$

$$\frac{d\mathbf{p}}{dt} = \oint_V \left(\underline{\mathbf{E}\,\mathrm{div}\,\mathbf{D} - [\mathbf{D} \times \mathrm{rot}\,\mathbf{E}]}\right)dv + \oint_V \left(\mathbf{H}\,\mathrm{div}\,\mathbf{B} - \underline{[\mathbf{B} \times \mathrm{rot}\,\mathbf{H}]}\right)dv -$$

$$- \oint_V [\partial\mathbf{D}/\partial t \times \mathbf{B}]\,dv - \oint_V [\mathbf{D} \times \partial\mathbf{B}/\partial t]\,dv$$

$$\oint_V \left(\mathbf{E}\,\mathrm{div}\,\mathbf{D} - [\mathbf{D} \times \mathrm{rot}\,\mathbf{E}]\right)dv = \varepsilon\varepsilon_0 \oint_V \left(\mathbf{E}\,\mathrm{div}\,\mathbf{E} - [\mathbf{E} \times \mathrm{rot}\,\mathbf{E}]\right)dv \Rightarrow$$

$$\Rightarrow \; = 0$$

Часть рисунка с идентификатором отношения rId11 не найдена в файле.

Воспользуемся двумя формулами

$$[\mathbf{E} \cdot \mathrm{rot}\,\mathbf{E}] = \frac{1}{2}\mathrm{grad}\,\mathbf{E}^2 - (\mathbf{E} \cdot \vec{\nabla})\mathbf{E}$$

$$\mathbf{E}\,\mathrm{div}\,\mathbf{E} = (\vec{\nabla}\cdot\mathbf{E})\mathbf{E} - (\mathbf{E}\cdot\vec{\nabla})\mathbf{E}$$

х-ая компонента

$$E_x\partial_x E_x + E_x\partial_y E_y + E_x\partial_z E_z =$$

$$= \partial_x E_x^2 + \partial_y E_y E_x + \partial_z E_z E_x - E_x\partial_x E_x - E_y\partial_y E_x - E_z\partial_z E_x$$

$$[\mathbf{E}\times\mathrm{rot}\,\mathbf{E}] = \frac{1}{2}\mathrm{grad}\,\mathbf{E}^2 - \underline{(\mathbf{E}\cdot\vec{\nabla})\mathbf{E}}$$

х-ая компонента

$$E_y(\mathrm{rot}\,\mathbf{E})_z - E_z(\mathrm{rot}\,\mathbf{E})_y = E_x\partial_x E_x + E_y\partial_x E_y + E_z\partial_x E_z -$$

$$-(E_x\partial_x + \underline{E_y\partial_y} + E_z\partial_z)E_x$$

$$\underline{E_y\partial_x E_y} - \underline{E_y\partial_y E_x} + \underline{E_z\partial_x E_z} - \underline{E_z\partial_z E_x} =$$

$$= E_x\partial_x E_x + \underline{E_y\partial_x E_y} + \underline{E_z\partial_x E_z} - E_x\partial_x E_x - \underline{E_y\partial_y E_x} - \underline{E_z\partial_z E_x}$$

$$\mathbf{E}\,\mathrm{div}\,\mathbf{E} - [\mathbf{E}\times\mathrm{rot}\,\mathbf{E}] = (\vec{\nabla}\cdot\mathbf{E})\mathbf{E} - \frac{1}{2}\vec{\nabla}\mathbf{E}^2$$

$$\mathbf{E}\,\mathrm{div}\,\mathbf{E} - [\mathbf{E}\times \mathrm{rot}\,\mathbf{E}] = (\vec{\nabla}\cdot\mathbf{E})\mathbf{E} - \frac{1}{2}\vec{\nabla}\mathbf{E}^2$$

$$\oint_V \left(\mathbf{E}\,\mathrm{div}\,\mathbf{E} - [\mathbf{E}\times \mathrm{rot}\,\mathbf{E}]\right)dv = \oint_V \{(\vec{\nabla}\cdot\mathbf{E})\mathbf{E} - \frac{1}{2}\vec{\nabla}\mathbf{E}^2\}dv =$$

$$E \sim \frac{1}{r^2}$$

$$\oint_V \vec{\nabla}\psi\, dv = \oint_S \mathbf{n}\psi\, dS$$

$$= \oint_S (\mathbf{n}\cdot\mathbf{E})\mathbf{E}\,dS - \frac{1}{2}\oint_S \mathbf{n}\cdot\mathbf{E}^2 dS = 0$$

$$\frac{d\mathbf{p}}{dt} = \oint_V \left(\mathbf{E}\,\mathrm{div}\,\mathbf{D} \underbrace{\cancel{- [\mathbf{D}\times \mathrm{rot}\,\mathbf{E}]}}_{=0}\right)dv + \oint_V \left(\mathbf{H}\,\mathrm{div}\,\mathbf{B} \underbrace{\cancel{- [\mathbf{B}\times \mathrm{rot}\,\mathbf{H}]}}_{=0}\right)dv -$$

$$- \oint_V [\partial\mathbf{D}/\partial t \times \mathbf{B}]dv - \oint_V [\mathbf{D}\times\partial\mathbf{B}/\partial t]dv = -\frac{\partial}{\partial t}\oint_V [\mathbf{D}\times\mathbf{B}]dv$$

$$\boxed{\frac{d\mathbf{p}}{dt} + \frac{\partial}{\partial t}\oint_V [\mathbf{D}\times\mathbf{B}]dv = 0 \qquad \mathbf{g} = [\mathbf{D}\times\mathbf{B}]}$$

- плотность импульса электромагнитного поля

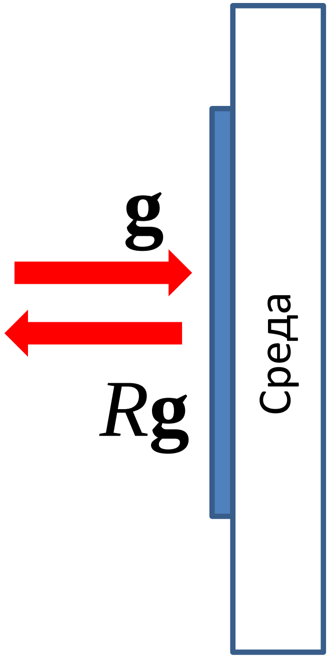
$$\mathbf{g} = [\mathbf{D} \times \mathbf{B}] = \varepsilon\varepsilon_0\mu\mu_0[\mathbf{E} \times \mathbf{H}] = \mathbf{S}_p / v^2$$

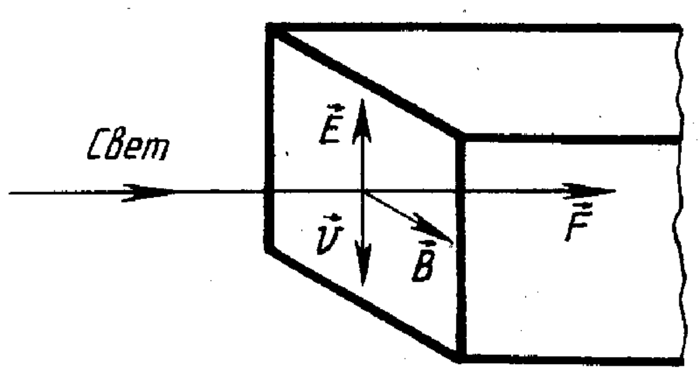
$$S_p v = w$$

$$w = (\mathbf{D} \cdot \mathbf{E}) / 2$$

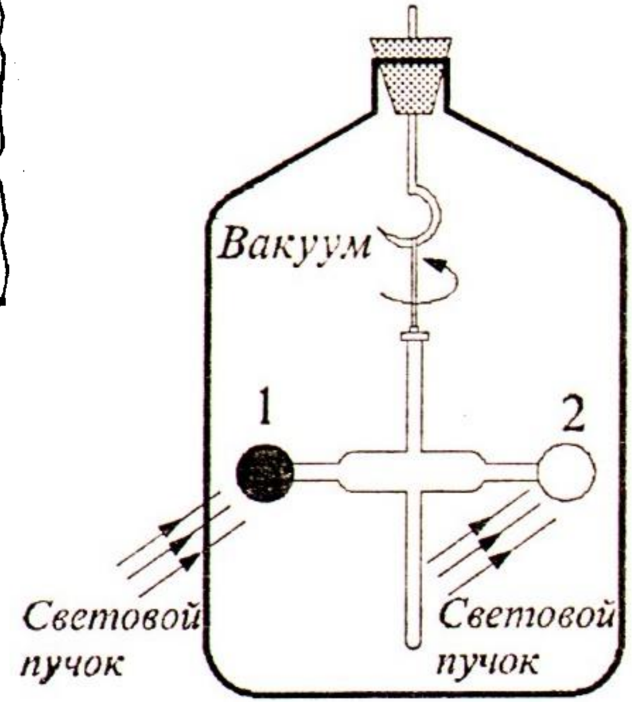
$$(1 + R) g \Delta S v \Delta t = p \Delta S \Delta t$$

$$p = (1 + R) g v = (1 + R) S_p / v = (1 + R) w / v^2$$





П.Н. Лебедев 1900



- 10^{14} Па - давление в центре взрыва водородной бомбы;
- 10^{13} Па - давление в центре Земли;
- $3 \cdot 10^{9}$ Па - давление колеса вагона на рельсы;
- $5 \cdot 10^{7}$ Па - давление жала пчелы;
- 10^{6} Па - давление конькобежца на лед;
- $4 \cdot 10^{5}$ Па - давление человека при ходьбе;
- 10^{-8} Па - давление воздуха на высоте 800 км.

Давление от лазерной указки 10^{-6} Па

Момент импульса (угловой момент) электромагнитной волны

$$\mathbf{L} = \int \mathbf{h} dv = \int [\mathbf{r}[\mathbf{D} \times \mathbf{B}]] dv$$

$$\mathbf{E} = \mathrm{Re}\{\mathbf{A}(\tilde{\mu}_1 z, \tilde{\mu}_2 t)\exp(-i\omega t + ikz)\}$$

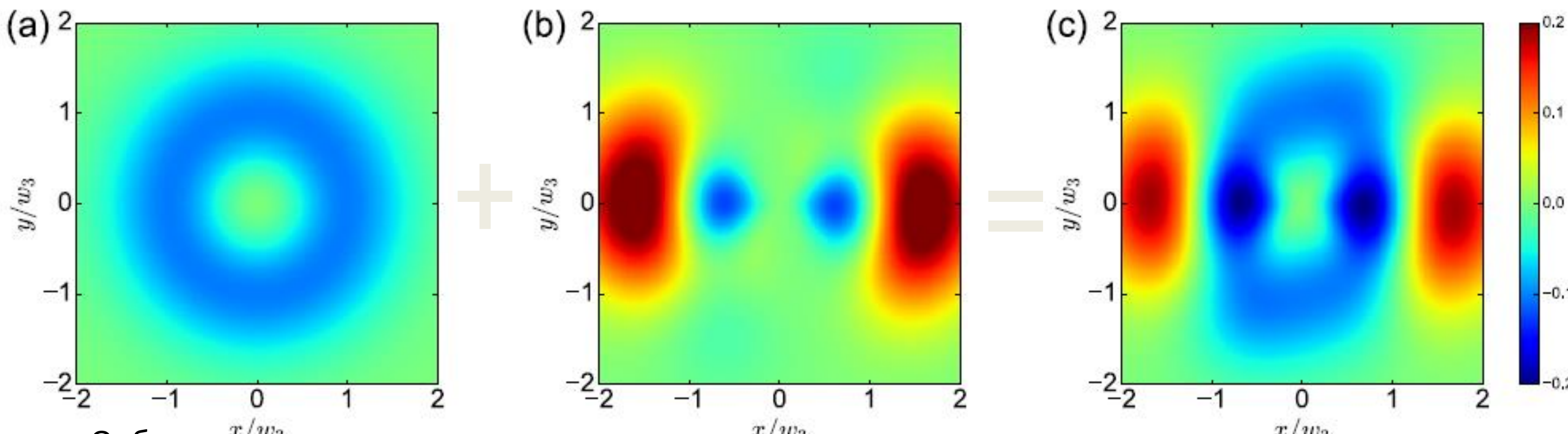
$$\mathbf{h} = \left\{\mathbf{r} \times \mathrm{Re}[i\mathbf{A} \times (\vec{\nabla} \times \mathbf{A}^*)]\right\}$$

$$L_z = \int h_z dv = -\frac{i}{16\pi\kappa}\int \sum_{j=x,y,z}\left(A_j^* \frac{\partial}{\partial \varphi} A_j\right) dv + \frac{1}{16\pi\kappa}\int \left(\left|A_+\right|^2 - \left|A_-\right|^2\right) dv$$

Плотность потока углового момента

«Орбитальная» часть углового момента

«Спиновая» часть углового момента



Орбитальная часть плотности потока углового момента

Спиновая часть плотности потока углового момента

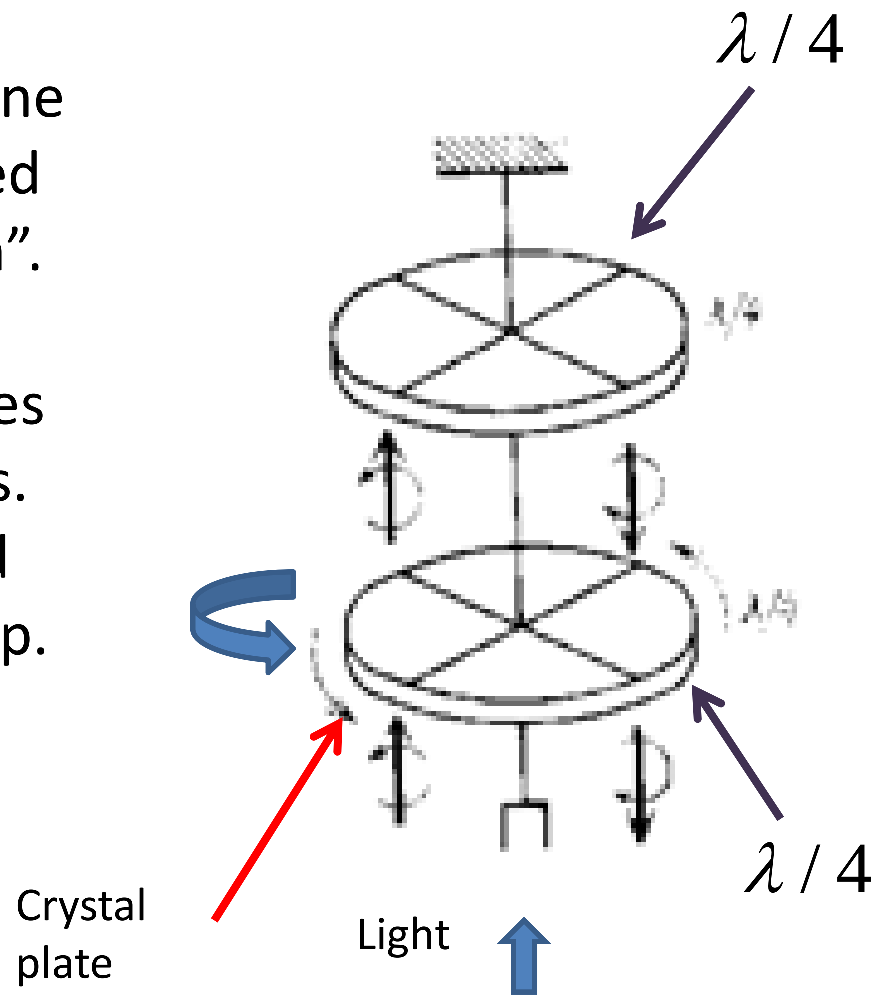
Плотность потока углового момента

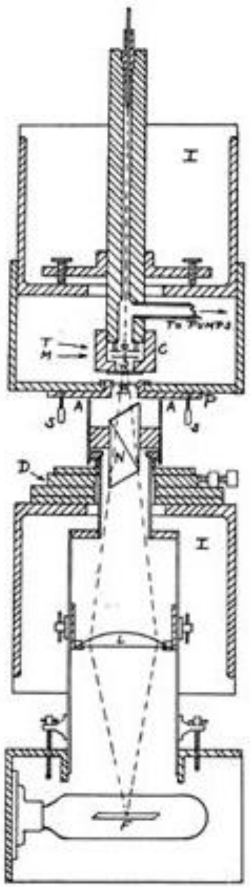
Угловой момент электромагнитной волны

$$\mathbf{L} = \int \mathbf{h}\, dv = \int [\mathbf{r}[\mathbf{D} \times \mathbf{B}]]\, dv$$

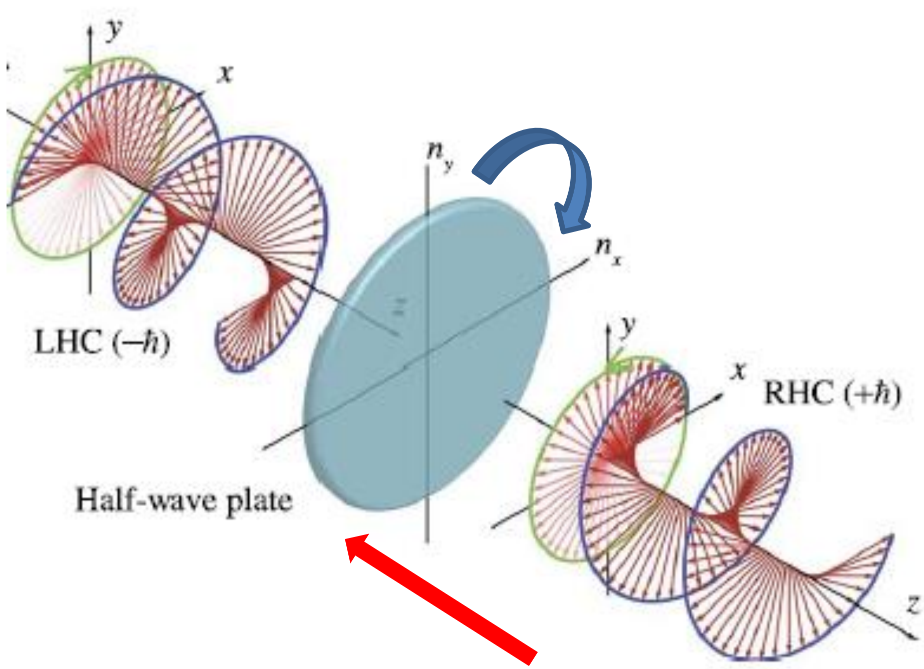
"... any apparatus changing the plane polarized light into a circle polarized should strive to come into rotation".

A.I. Sadovsky. Ponderomotive forces of electromagnetic and light waves. Journal of the Russian Physical and Chemical Society, 1897, v. 29, N 2, p. 82.

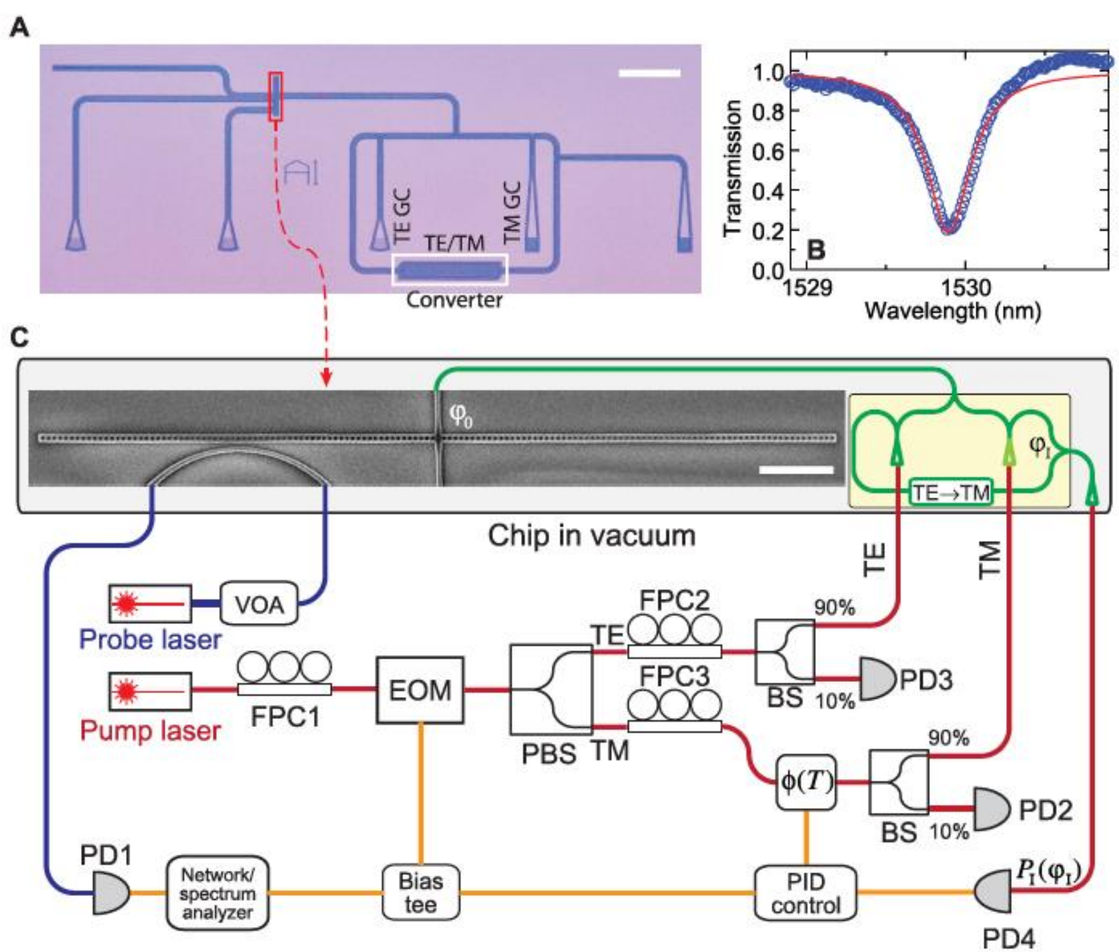




R.A. Beth. Mechanical detection of the angular momentum of light. Phys. Rev., v. 50, p. 115 (1936)



Beth showed that when linearly polarized light is converted to circularly polarized one by doubly refracting slab, the slab experience a reaction torque



"We demonstrate the measurement of the spin angular momentum of photons propagating in a birefringent waveguide and we use of optical torque to activate rotational motion of an optomechanical device. We show that the sign and magnitude of the optical torque are determined by the photons polarization states".

He L., Li H., Li M. *Sci. Adv.*, **2**, e1600485 (2016).

Квазистационарное приближение

$$\operatorname{rot}\mathbf{E}(\mathbf{r},t) = -\frac{\partial\mathbf{B}(\mathbf{r},t)}{\partial t}$$

$$\operatorname{rot}\mathbf{H}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) + \cancel{\frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t}} = 0$$

$$\operatorname{div}\mathbf{D}(\mathbf{r},t) = \rho(\mathbf{r},t)$$

$$\operatorname{div}\mathbf{B}(\mathbf{r},t) = 0$$

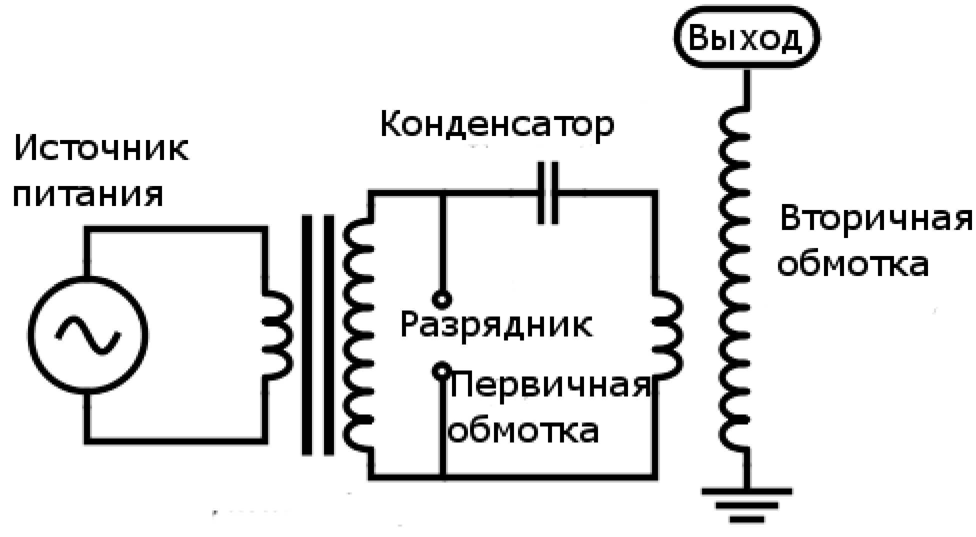
$$\mathbf{E} = \mathbf{E}_0 \sin\omega t \qquad \frac{\partial\mathbf{D}}{\partial t} = \varepsilon\varepsilon_0\omega\mathbf{E}_0\cos\omega t \qquad \mathbf{j} = \sigma\mathbf{E} = \sigma\mathbf{E}_0\sin\omega t$$

$$\mathbf{j}(\mathbf{r},t) >> \frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t} \Longleftrightarrow \sigma\mathbf{E}_0 >> \varepsilon\varepsilon_0\omega\mathbf{E}_0 \Rightarrow \sigma >> \varepsilon\varepsilon_0\omega$$

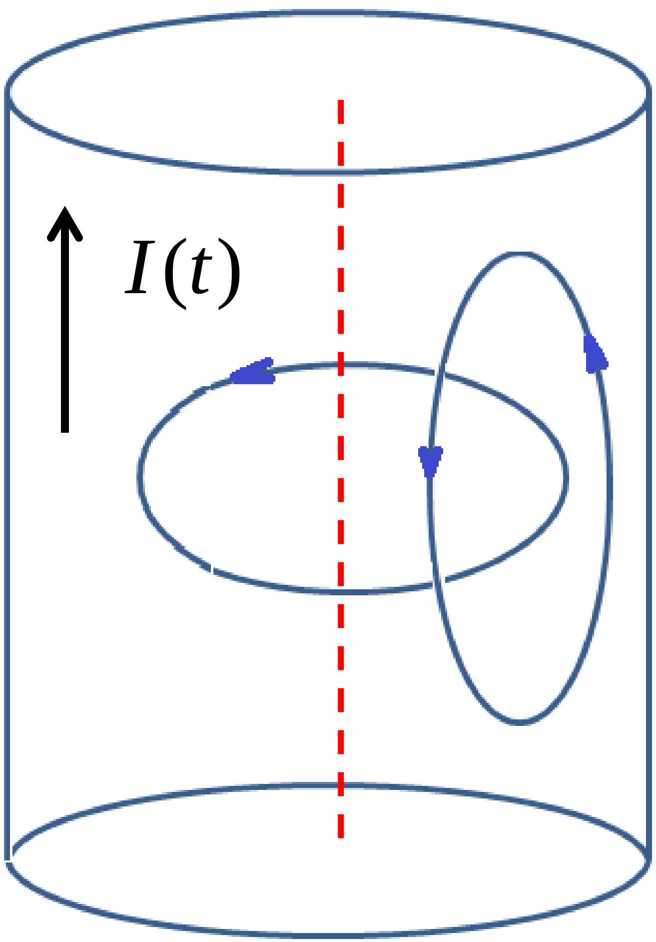
Для меди $\omega << 10^{17}\,\text{c}^{-1}$

Току, меняющемуся с частотой 50 Гц $\Rightarrow$ $\lambda \approx 600$ км $>> \mathrm{L}$

$$\operatorname{rot}\mathbf{H}(\mathbf{r},t)=\mathbf{j}(\mathbf{r},t)+\frac{\partial\mathbf{D}(\mathbf{r},t)}{\partial t}$$



Скин-эффект



$$\frac{dI(t)}{dt} > 0 \qquad \operatorname{rot}\mathbf{E}(\mathbf{r},t) = -\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t}$$

$$\operatorname{rot}\mathbf{H}(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) = \sigma\mathbf{E}$$

$$\operatorname{rotrot}\mathbf{E}(\mathbf{r},t) = -\operatorname{rot}\frac{\partial \mathbf{B}(\mathbf{r},t)}{\partial t} =$$

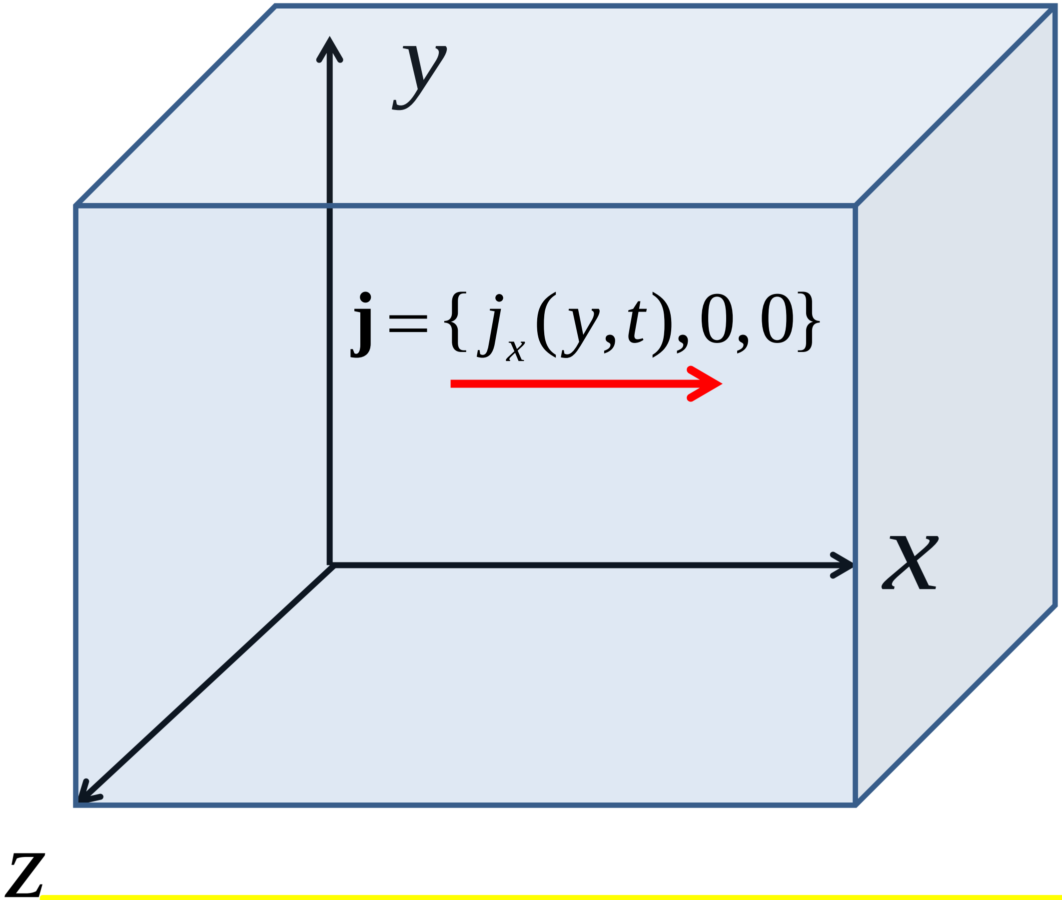
$$= -\frac{\partial}{\partial t}\operatorname{rot}\mathbf{B}(\mathbf{r},t) = -\mu\mu_0\frac{\partial}{\partial t}\operatorname{rot}\mathbf{H}(\mathbf{r},t) =$$

$$= -\mu\mu_0\sigma\frac{\partial \mathbf{E}(\mathbf{r},t)}{\partial t}$$

$$\operatorname{rotrot}\mathbf{E}(\mathbf{r},t) = \underbrace{\operatorname{graddiv}\mathbf{E}(\mathbf{r},t)}_{=0} - \Delta\mathbf{E}(\mathbf{r},t)$$

$$\operatorname{div}\mathbf{D} = 0$$

$$\Delta\mathbf{E} = \mu\mu_0\sigma\frac{\partial \mathbf{E}}{\partial t}$$



$$\mathbf{E}=\{E_x(y,t),0,0\}$$

$$E_x(y,t)=E_0(y)\exp(i\omega t)$$

$$\Delta\mathbf{E}=\mu\mu_0\sigma\frac{\partial\mathbf{E}}{\partial t}$$

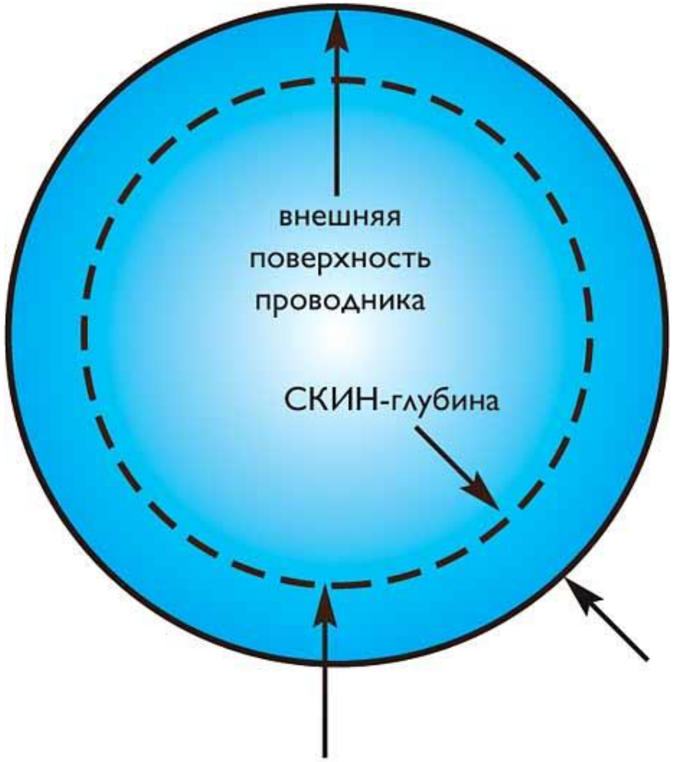
$$\frac{d^2E_0}{dy^2}=i\mu\mu_0\sigma\omega E_0$$

$E_0=A\exp[-\alpha(1+i)y]$ где $\alpha=(\mu\mu_0\omega\sigma/2)^{1/2}$

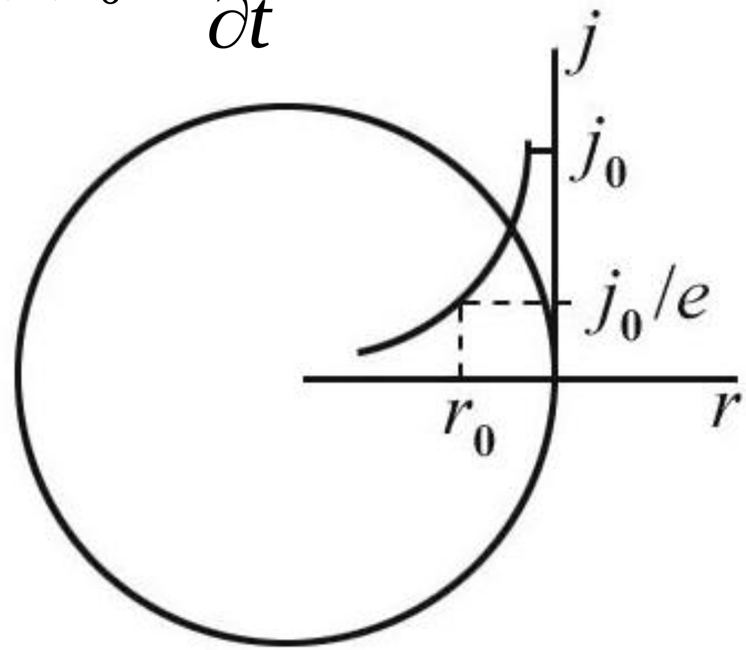
$j_x=j_0\exp[-\alpha(1+i)y]$ $d=1/\alpha$ - толщина скин-слоя

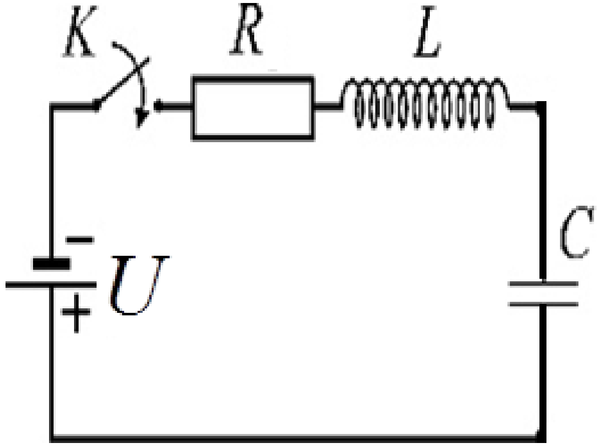
Для меди при $\omega=10^4\,\text{с}^{-1}$ $d=4$ мм

$$\Delta \mathbf{E} = \mu\mu_0\sigma\frac{\partial \mathbf{E}}{\partial t}$$



Плотность тока здесь в А/м² в 1/2,72 раза меньше, чем на поверхности





$$U - L\frac{dI}{dt} = IR + \frac{q}{C} \qquad I = \frac{dq}{dt}$$

$$q(0) = 0 \qquad I(0) = 0$$

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{U}{L}$$

$$\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=\frac{U}{L}$$

$$q(t)=CU+A\exp\left(-\frac{R}{2L}t\right)\sin\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\times t+\phi\right)$$

$$I(t)=\frac{dq}{dt}=A\exp\left(-\frac{R}{2L}t\right)\left\{-\frac{R}{2L}\sin\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\times t+\phi\right)+\right.$$

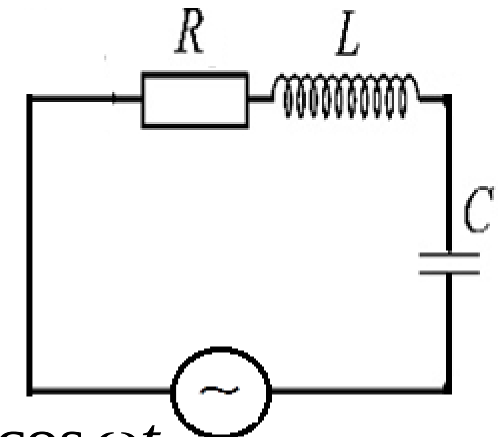
$$\left.+\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\cos\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\times t+\phi\right)\right\}$$

$$0=CU+A\sin\phi \qquad 0=\left(-\frac{R}{2L}\right)\sin\phi+\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\cos\phi$$

$$A=-\frac{CU}{\sqrt{1-CR^2/4L}} \qquad \phi=\mathrm{arctg}\sqrt{\frac{4L}{CR^2}-1}$$

$$q(t) = CU\left[1 - \frac{\exp\left(-Rt/2L\right)}{\sqrt{1 - CR^2/4L}}\sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \times t + \operatorname{arctg}\sqrt{\frac{4L}{CR^2} - 1}\right)\right]$$

Метод комплексных амплитуд



$U_0 \cos \omega t$

$$\frac{d^2 q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{q}{LC} = \frac{U_0}{L}\cos \omega t$$

$$L\frac{d^2 I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = -\omega U_0 \sin \omega t$$

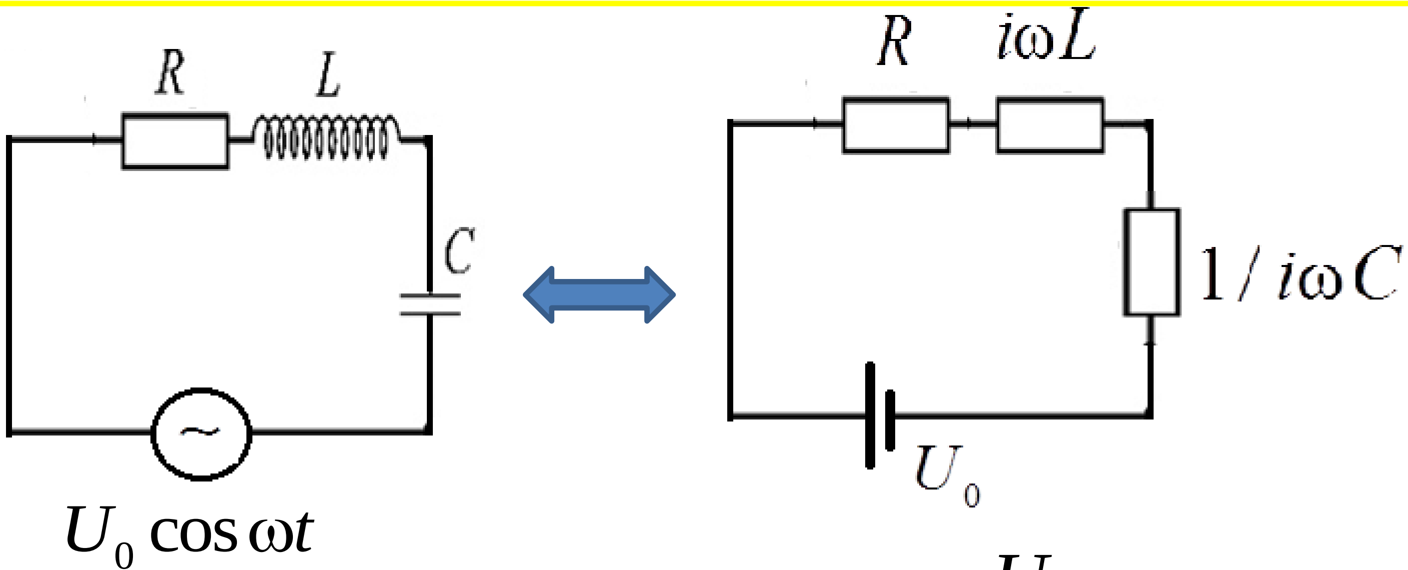
$$U_0 \cos \omega t = U_0 \operatorname{Re}\{\exp(i\omega t)\} \qquad I(t) = \operatorname{Re}\{\hat{I}\exp(i\omega t)\}$$

$$\operatorname{Re}\left\{\left[L\frac{d^2}{dt^2} + R\frac{d}{dt} + \frac{1}{C}\right]\hat{I}\exp(i\omega t)\right\} = U_0 \operatorname{Re}\{i\omega \exp(i\omega t)\}$$

$$\left[L\frac{d^2}{dt^2}+R\frac{d}{dt}+\frac{1}{C}\right]\hat{I}\exp(i\omega t)=U_0\{i\omega\exp(i\omega t)\}$$

$$\left[-\omega^2L+i\omega R+\frac{1}{C}\right]\hat{I}=i\omega U_0 \qquad \left[i\omega L+R+\frac{1}{i\omega C}\right]\hat{I}=U_0$$

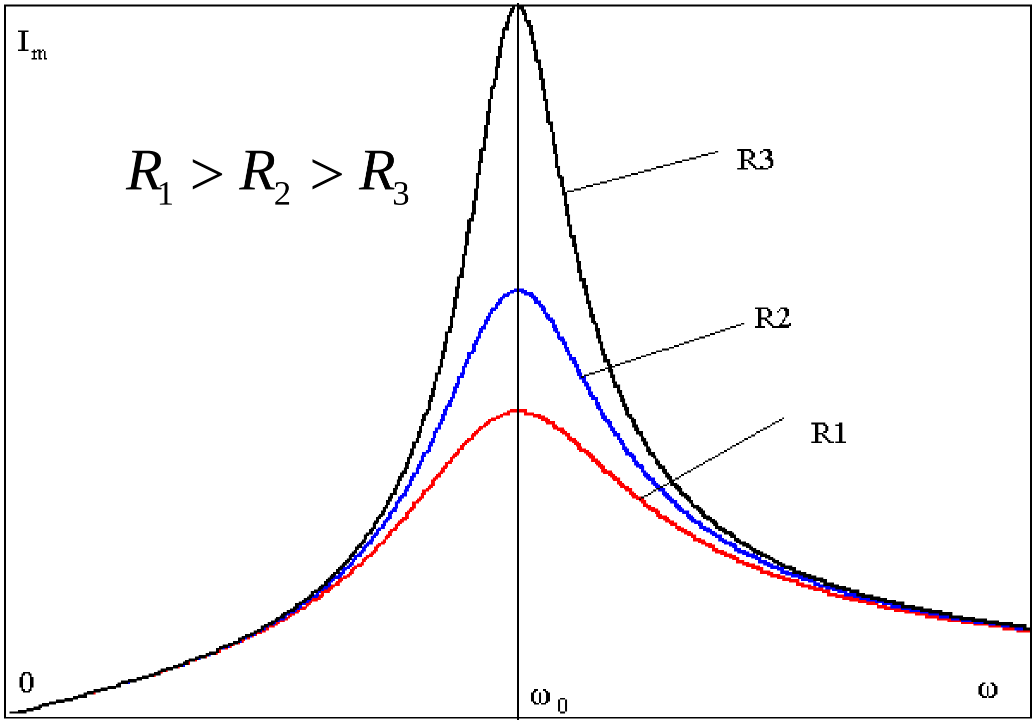
$$L\Rightarrow Z_L=i\omega L \quad R\Rightarrow Z_R=R \quad C\Rightarrow Z_C=1/i\omega C$$



$$\left[Z_L+Z_R+Z_C\right]\hat{I}=U_0 \qquad \hat{I}=\frac{U_0}{Z_L+Z_R+Z_C}$$

$$I(t) = \mathrm{Re}\left\{\hat{I}\exp(i\omega t)\right\} = \mathrm{Re}\left\{\frac{U_0 \exp(i\omega t)}{i\omega L + R + 1/i\omega C}\right\}$$

$$I(t) = \frac{U_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}} \cos\left(\omega t - \mathrm{arctg}\left(\frac{\omega L - 1/\omega C}{R}\right)\right)$$



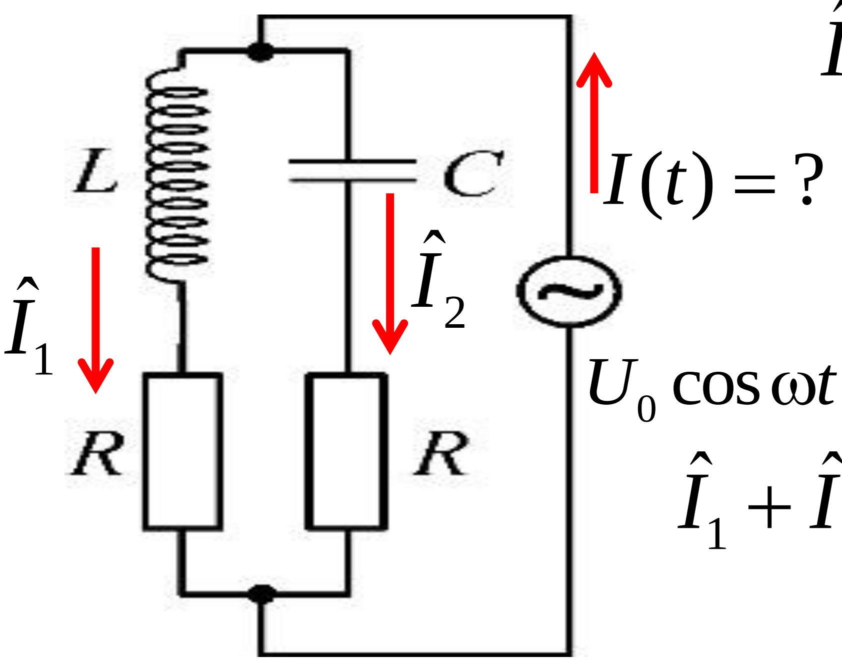
$$I_m = \frac{U_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$\Phi = \mathrm{arctg}\left(\frac{\omega L - 1/\omega C}{R}\right)$$

$$\omega_0 \approx \omega_k = \sqrt{1/LC}$$

$$\Phi(\omega_m \approx \omega_0) = 0$$

Законы Кирхгофа и метод комплексных амплитуд



$$\hat{I}_1 = \frac{U_0}{R + i\omega L} \qquad \hat{I}_2 = \frac{U_0}{R + 1/i\omega C}$$

$$I(t) = \mathrm{Re}\left\{(\hat{I}_1 + \hat{I}_2)\exp(i\omega t)\right\}$$

$$\hat{I}_1 + \hat{I}_2 = U_0 \frac{2R + i(\omega L - 1/\omega C)}{(R + i\omega L)(R + 1/i\omega C)}$$

$$I(t) = U_0 \frac{\sqrt{4R^2 + (\omega L - 1/\omega C)^2}}{\sqrt{(R^2 + L/C)^2 + R^2(\omega L - 1/\omega C)^2}} \times$$

$$\times \cos\left\{\omega t - \mathrm{arctg}(\omega L/R) + \mathrm{arctg}(1/\omega RC) + \mathrm{arctg}\left(\frac{\omega L - 1/\omega C}{2R}\right)\right\}$$

Функции Лагранжа и Гамильтона частицы движущейся в электромагнитном поле

$$m\frac{d\mathbf{v}}{dt} = \frac{d\mathbf{p}}{dt} = q\mathbf{E}(\mathbf{r}) + q[\mathbf{v}\times\mathbf{B}(\mathbf{r})]$$

$$\boxed{\begin{gathered} L(\mathbf{r},\mathbf{v}) = m\mathbf{v}^2/2 - q\varphi(\mathbf{r}) + q\mathbf{A}\cdot\mathbf{v} \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \left(\frac{\partial L}{\partial \mathbf{r}}\right) = 0 \end{gathered}}$$

$\mathbf{P} = \dfrac{\partial L}{\partial \mathbf{v}} = m\mathbf{v} + q\mathbf{A}$ - обобщенный импульс

$$\frac{\partial L}{\partial \mathbf{r}} = -q\,\mathrm{grad}\,\varphi + q\,\mathrm{grad}(\mathbf{A}\cdot\mathbf{v}) = -q\,\mathrm{grad}\,\varphi + q(\mathbf{v}\cdot\mathrm{grad})\mathbf{A} +$$

$$+\cancel{q(\mathbf{A}\cdot\mathrm{grad})\mathbf{v}} + q[\mathbf{v}\times\mathrm{rot}\,\mathbf{A}] + \cancel{q[\mathbf{A}\times\mathrm{rot}\,\mathbf{v}]} = -q\,\mathrm{grad}\,\varphi +$$

$$+q(\mathbf{v}\cdot\mathrm{grad})\mathbf{A} + q[\mathbf{v}\times\mathbf{B}]$$

$$\frac{\partial L}{\partial \mathbf{r}} = -q\,\mathrm{grad}\,\varphi + q(\mathbf{v}\cdot\mathrm{grad})\mathbf{A} + q[\mathbf{v}\times\mathbf{B}]$$

$$\boxed{\begin{aligned}&L(\mathbf{r},\mathbf{v}) = \\ &= m\mathbf{v}^2/2 - q\varphi(\mathbf{r}) + q\mathbf{A}\cdot\mathbf{v}\end{aligned}}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) = \frac{d}{dt}(m\mathbf{v}+q\mathbf{A}) = m\frac{d^2\mathbf{r}}{dt^2} + q\frac{\partial \mathbf{A}}{\partial t} +$$

$$+q\left(\frac{\partial \mathbf{A}}{\partial x}v_x + \frac{\partial \mathbf{A}}{\partial y}v_y + \frac{\partial \mathbf{A}}{\partial z}v_z\right) = m\frac{d^2\mathbf{r}}{dt^2} + q\frac{\partial \mathbf{A}}{\partial t} + q(\mathbf{v}\cdot\mathrm{grad})\mathbf{A}$$

$$0 = \frac{d}{dt}\left(\frac{\partial L}{\partial \mathbf{v}}\right) - \left(\frac{\partial L}{\partial \mathbf{r}}\right) = m\frac{d^2\mathbf{r}}{dt^2} + q\frac{\partial \mathbf{A}}{\partial t} + \cancel{q(\mathbf{v}\cdot\mathrm{grad})\mathbf{A}} + q\,\mathrm{grad}\,\varphi -$$

$$-\cancel{q(\mathbf{v}\cdot\mathrm{grad})\mathbf{A}} - q[\mathbf{v}\times\mathbf{B}]$$

$$m\frac{d^2\mathbf{r}}{dt^2} = q\underbrace{\left\{-\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\,\varphi\right\}}_{=\mathbf{E}} + q[\mathbf{v}\times\mathbf{B}] = q\mathbf{E} + q[\mathbf{v}\times\mathbf{B}]$$

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{P}}, \qquad \frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{r}}, \qquad H(\mathbf{r},\mathbf{P}) = \frac{1}{2m}(\mathbf{P} - q\mathbf{A})^2 + q\varphi$$

$$-\partial H / \partial \mathbf{r} = -q\,\mathrm{grad}\,\varphi - \frac{1}{2m}\mathrm{grad}\{(\mathbf{P} - q\mathbf{A})\cdot(\mathbf{P} - q\mathbf{A})\} =$$

$$\mathrm{grad}(\mathbf{a}\cdot\mathbf{a}) = 2[\mathbf{a}\times \mathrm{rot}\,\mathbf{a}] + 2(\mathbf{a}\cdot \mathrm{grad})\mathbf{a}$$

$$-\partial H / \partial \mathbf{r} = -q\,\mathrm{grad}\,\varphi - \frac{1}{m}\left[(\mathbf{P} - q\mathbf{A})\times \mathrm{rot}(\mathbf{P} - q\mathbf{A})\right] -$$

$$-\frac{1}{m}\big((\mathbf{P} - q\mathbf{A})\cdot \mathrm{grad}\big)(\mathbf{P} - q\mathbf{A}) =$$

$$\mathbf{P} = m\mathbf{v} + q\mathbf{A}$$

$$\mathrm{rot}\,\mathbf{P} = 0$$

$$\mathrm{grad}\,\mathbf{P} = 0$$

$$= -q\,\mathrm{grad}\,\varphi + \frac{q}{m}[m\mathbf{v}\times \mathrm{rot}\,\mathbf{A}] + \frac{q}{m}(m\mathbf{v}\cdot \mathrm{grad})\mathbf{A} =$$

$$= -q\,\mathrm{grad}\,\varphi + q[\mathbf{v}\times\mathbf{B}] + q(\mathbf{v}\cdot \mathrm{grad})\mathbf{A} \Rightarrow$$

$$\mathbf{E} = -\operatorname{grad}\varphi - \frac{\partial \mathbf{A}}{\partial t}$$

$$-\partial H/\partial \mathbf{r} = -q\operatorname{grad}\varphi + q[\mathbf{v}\times\mathbf{B}] + q(\mathbf{v}\cdot\operatorname{grad})\mathbf{A} =$$

$$= q\mathbf{E} + q\frac{\partial \mathbf{A}}{\partial t} + q[\mathbf{v}\times\mathbf{B}] + q(\mathbf{v}\cdot\operatorname{grad})\mathbf{A} =$$

$$= m\frac{d^2\mathbf{r}}{dt^2} + q\frac{\partial \mathbf{A}}{\partial t} + q(\mathbf{v}\cdot\operatorname{grad})\mathbf{A}$$

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt}(m\mathbf{v} + q\mathbf{A}) = m\frac{d^2\mathbf{r}}{dt^2} + q\frac{\partial \mathbf{A}}{\partial t} + q(\mathbf{v}\cdot\operatorname{grad})\mathbf{A}$$

$$\boxed{\frac{d\mathbf{P}}{dt} = -\frac{\partial H}{\partial \mathbf{r}},}$$

$$H(\mathbf{r},\mathbf{P}) = \frac{1}{2m}(\mathbf{P} - q\mathbf{A})^2 + q\varphi$$

$$\mathbf{P} = m\mathbf{v} + q\mathbf{A}$$

$$\frac{\partial H}{\partial \mathbf{P}} = \frac{(\mathbf{P} - q\mathbf{A})}{m} = \frac{m\mathbf{v}}{m} = \frac{d\mathbf{r}}{dt}$$

$$\boxed{\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{P}}}$$

Система единиц СИ и система единиц Гаусса

$$\mathbf{F}_{12} = \frac{q_1 q_2}{r_{12}^3}\mathbf{r}_{12} \Rightarrow d\mathbf{F}_{12} = \alpha \frac{I_1 I_2 [d\mathbf{l}_2 \times [d\mathbf{l}_1 \times \mathbf{r}_{12}]]}{r_{12}^3}$$

$$\mathbf{F}_{12} = \beta \frac{q_1 q_2}{r_{12}^3}\mathbf{r}_{12} \Leftarrow d\mathbf{F}_{12} = \frac{I_1 I_2 [d\mathbf{l}_2 \times [d\mathbf{l}_1 \times \mathbf{r}_{12}]]}{r_{12}^3}$$

$$\frac{\beta}{\alpha} = \frac{1}{c^2}$$

Система единиц СИ и система единиц Гаусса

$$j \Rightarrow (4\pi\varepsilon_0)^{1/2} j \quad Q \Rightarrow (4\pi\varepsilon_0)^{1/2} Q \quad \rho \Rightarrow (4\pi\varepsilon_0)^{1/2} \rho \quad \varepsilon \Rightarrow \varepsilon\varepsilon_0$$

$$\sigma \Rightarrow (4\pi\varepsilon_0)\sigma \quad C \Rightarrow (4\pi\varepsilon_0)C \quad E \Rightarrow (4\pi\varepsilon_0)^{-1/2} E \quad \mu \Rightarrow \mu\mu_0$$

$$D \Rightarrow (\varepsilon_0 / 4\pi)^{1/2} D \quad H \Rightarrow (4\pi\mu_0)^{-1/2} H \quad B \Rightarrow (\mu_0 / 4\pi)^{1/2} B$$

$$\Phi \Rightarrow (\mu_0 / 4\pi)^{1/2} \Phi \quad L \Rightarrow (4\pi\varepsilon_0)^{-1} L \quad P \Rightarrow (4\pi\varepsilon_0) P$$

$$R \Rightarrow (4\pi\varepsilon_0)^{-1} R \quad A \Rightarrow (\mu_0 / 4\pi)^{1/2} A \quad \varphi \Rightarrow (4\pi\varepsilon_0)^{-1/2} \varphi$$

$$\mathrm{rot}\,\mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\mathrm{rot}(4\pi\mu_0)^{-1/2}\mathbf{H} = (4\pi\varepsilon_0)^{1/2}\mathbf{j} + \frac{\partial(\varepsilon_0 / 4\pi)^{1/2}\mathbf{D}}{\partial t}$$

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

Уравнения Максвелла в системе единиц Гаусса

$$\mathrm{rot}\,\mathbf{H} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}$$

$$\mathrm{rot}\,\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\mathrm{div}\,\mathbf{D} = 4\pi\rho$$

$$\mathrm{div}\,\mathbf{B} = 0$$

$$\mathbf{B} = \mu\mathbf{H}$$

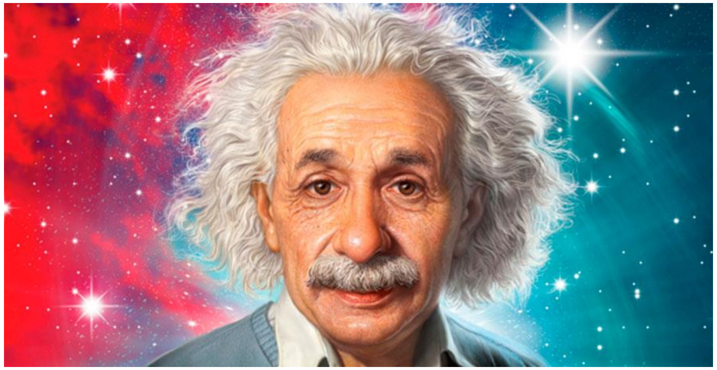
$$\mathbf{D} = \varepsilon\mathbf{E}$$

Здесь *c* – скорость света в некоторой конкретной инерциальной системе координат. А как уравнения Максвелла будут выглядеть в системе координат, движущейся относительно нее с постоянной скоростью?

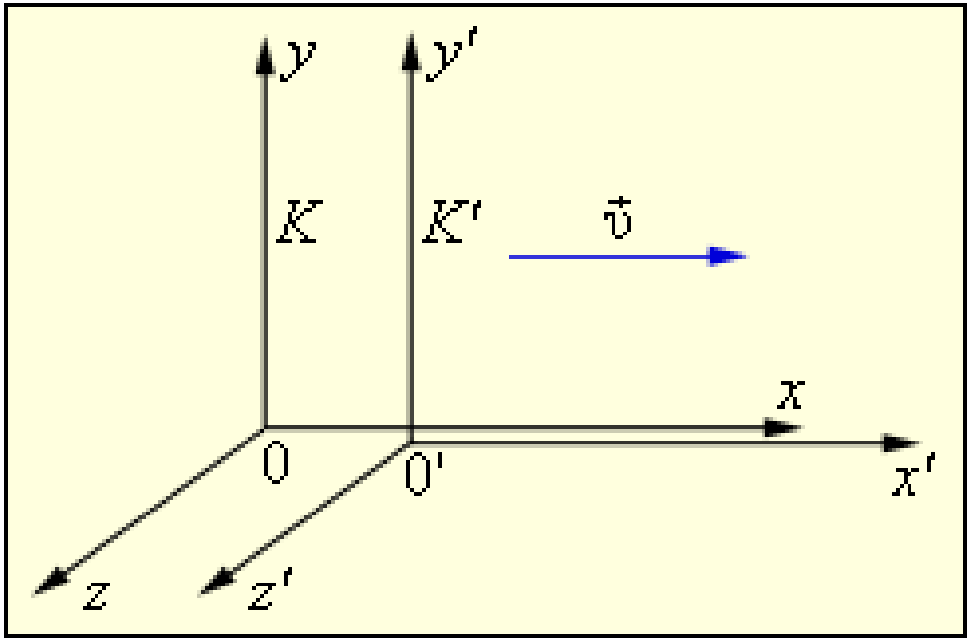
Постулаты специальной теории относительности

1. Все законы природы одинаковы в инерциальных системах отсчета (равномерное прямолинейное движение не оказывает влияния на физические процессы и в любой инерциальной системе координат возникает одна и та же функциональная зависимость между величинами).

2. Любые взаимодействия между телами распространяются в пустоте с универсальной конечной скоростью, равной скорости света в пустоте, одинаковой во всех инерциальных системах отсчета.



А. Эйнштейн



$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y \qquad z' = z$$

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

или $$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \qquad y = y' \qquad z = z' \qquad t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}$$

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \qquad u_y = \frac{u'_y\sqrt{1 - v^2/c^2}}{1 + vu'_x/c^2} \qquad u_z = \frac{u'_z\sqrt{1 - v^2/c^2}}{1 + vu'_x/c^2}$$

$$\frac{d}{dt}\frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}} = \mathbf{F} \qquad \frac{d}{dt}\frac{mc^2}{\sqrt{1 - v^2/c^2}} = \mathbf{F}\cdot\mathbf{v}$$

v - скорость частицы

Назовем четырехвектором a_α совокупность четырех величин, которые при преобразовании Лоренца преобразуются по закону $a_\alpha = \gamma_{\alpha\beta} a'_\beta$, а четырехтензором второго ранга $A_{\alpha\beta}$ – совокупность шестнадцати величин, таких, которые при указанном преобразовании изменяются по формуле $A_{\alpha\beta} = \gamma_{\alpha\delta}\gamma_{\beta\mu} A'_{\delta\mu}$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dfrac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{-iv/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{iv/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}$$

Четырехвектор $r_\alpha = \{x, y, z, ict\}$ $\alpha = 1, 2, 3, 4$

$$r_\alpha r_\alpha = x^2 + y^2 + z^2 - c^2t^2 = 0 = \text{const}$$

Переход от x', y', z', t' к x, y, z, t полностью эквивалентен повороту вектора r_α в четырехмерном пространстве с матрицей перехода

$$r_\alpha = \gamma_{\alpha\beta} r'_\beta$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{-iv/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{iv/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}$$

$$x = \gamma_{11} x' + \gamma_{14} ict'$$

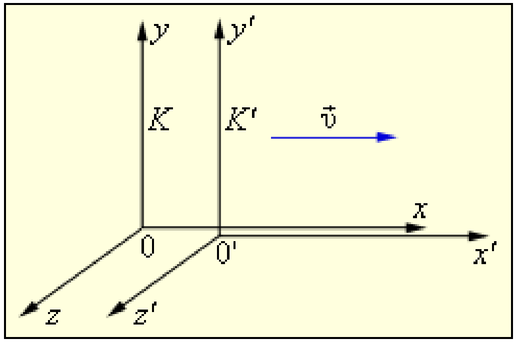
$$\Downarrow$$

$$x = \frac{x' + vt'}{\sqrt{1-v^2/c^2}}$$

$$ict = \gamma_{41} x' + \gamma_{44} ict' = \frac{iv/c}{\sqrt{1-v^2/c^2}} x' + \frac{1}{\sqrt{1-v^2/c^2}} ict' \Rightarrow t = \frac{t' + vx'/c^2}{\sqrt{1-v^2/c^2}}$$

Обратное преобразование $r'_\alpha = \tilde{\gamma}_{\alpha\beta} r_\beta$

$$\tilde{\gamma}_{\alpha\beta} = \begin{pmatrix} \dfrac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{iv/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{-iv/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}$$



Свойства матрицы $\gamma_{\alpha\beta}$ $\Rightarrow$ $\boxed{\gamma_{\alpha\beta}\gamma_{\alpha\rho} = \delta_{\beta\rho}}$

Квадрат длины четырехвектора $r_\alpha = \{x, y, z, ict\}$ инвариант

$$r_\alpha^2 = \gamma_{\alpha\beta} r'_\beta \, \gamma_{\alpha\rho} r'_\rho = \delta_{\beta\rho} r'_\beta \, r'_\rho = {r'_\beta}^2 = {r'_\alpha}^2$$

Четырехвекторы в релятивистской электродинамике

Четырехвектор тока $j_\alpha = \{j_x, j_y, j_z, ic\rho\}$

Уравнение непрерывности $\dfrac{\partial\rho}{\partial t} + \operatorname{div}\mathbf{j} = 0$ запишется в релятивистски инвариантном виде $\boxed{\partial j_\alpha / \partial r_\alpha = 0}$

$$r_\mu = \gamma_{\mu\alpha} r'_\alpha \qquad j'_\alpha = \tilde{\gamma}_{\alpha\beta} j_\beta \qquad \partial / \partial r'_\alpha = \partial / \partial r_\mu \; \gamma_{\mu\alpha}$$

$$\frac{\partial}{\partial r'_\alpha} j'_\alpha = \frac{\partial}{\partial r_\mu} \gamma_{\mu\alpha} \; \tilde{\gamma}_{\alpha\beta} j_\beta = \frac{\partial}{\partial r_\mu} \delta_{\mu\beta} j_\beta = \frac{\partial}{\partial r_\beta} j_\beta$$

$$j_\alpha = \gamma_{\alpha\beta} j'_\beta \qquad j_x = \frac{j'_x + v\rho'}{\sqrt{1 - v^2/c^2}} \qquad j_y = j'_y \qquad j_z = j'_z$$

$$ic\rho = \frac{j'_x (iv/c)}{\sqrt{1 - v^2/c^2}} + \frac{ic\rho'}{\sqrt{1 - v^2/c^2}} \qquad \rho = \frac{j'_x (v/c^2) + \rho'}{\sqrt{1 - v^2/c^2}}$$

$$\Delta\mathbf{A} - \mu\mu_0\varepsilon\varepsilon_0 \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu\mu_0 \mathbf{j}(\mathbf{r}, t)$$

$$\Delta\varphi - \mu\mu_0\varepsilon\varepsilon_0 \frac{\partial^2 \varphi}{\partial t^2} = -\rho(\mathbf{r}, t) / \varepsilon\varepsilon_0$$

$A \Rightarrow (\mu_0 / 4\pi)^{1/2} A$ $\quad j \Rightarrow (4\pi\varepsilon_0)^{1/2} j$ $\quad \varphi \Rightarrow (4\pi\varepsilon_0)^{-1/2} \varphi$ $\quad \rho \Rightarrow (4\pi\varepsilon_0)^{1/2} \rho$

$$\boxed{\Delta\mathbf{A} - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\frac{4\pi}{c} \mathbf{j}(\mathbf{r}, t)} \qquad \boxed{\Delta\varphi - \frac{\varepsilon\mu}{c^2} \frac{\partial^2 \varphi}{\partial t^2} = -4\pi\rho(\mathbf{r}, t)}$$

Четырехвектор потенциала $\quad A_\alpha = \{A_x, A_y, A_z, i\varphi\}$

$$\boxed{\frac{\partial^2 A_\alpha}{\partial r_\mu^2} = -\frac{4\pi}{c} j_\alpha}$$

(записано в релятивистски инвариантной форме в вакууме)

$$\frac{\partial^2 A'_\alpha}{\partial r'^2_\mu} = -\frac{4\pi}{c} j'_\alpha$$

$$\partial / \partial r'_\mu = \partial / \partial r_m \ \gamma_{m\mu} \qquad A'_\alpha = \tilde{\gamma}_{\alpha\beta} A_\beta \quad j'_\alpha = \tilde{\gamma}_{\alpha\beta} j_\beta \quad r_\mu = \gamma_{\mu\alpha} r'_\alpha$$

$$\frac{\partial}{\partial r_m} \gamma_{m\mu} \frac{\partial}{\partial r_n} \gamma_{n\mu} \tilde{\gamma}_{\alpha\beta} A_\beta = -\frac{4\pi}{c} \tilde{\gamma}_{\alpha\beta} j_\beta$$

$$\frac{\partial}{\partial r_m} \frac{\partial}{\partial r_n} \delta_{mn} \tilde{\gamma}_{\alpha\beta} A_\beta = \frac{\partial^2}{\partial r_m^2} \tilde{\gamma}_{\alpha\beta} A_\beta = -\frac{4\pi}{c} \tilde{\gamma}_{\alpha\beta} j_\beta$$

$$\gamma_{\sigma\alpha} \frac{\partial^2}{\partial r_m^2} \tilde{\gamma}_{\alpha\beta} A_\beta = -\frac{4\pi}{c} \gamma_{\sigma\alpha} \tilde{\gamma}_{\alpha\beta} j_\beta \qquad \gamma_{\sigma\alpha} \tilde{\gamma}_{\alpha\beta} = \delta_{\sigma\beta}$$

$$\frac{\partial^2}{\partial r_m^2} A_\sigma = -\frac{4\pi}{c} j_\sigma$$

Преобразование потенциала $A_\alpha = \{A_x, A_y, A_z, i\varphi\}$

$$A_\alpha = \gamma_{\alpha\beta} A'_\beta$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dfrac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{-iv/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{iv/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}$$

$$A_x = \frac{A'_x + (v/c)\varphi'}{\sqrt{1-v^2/c^2}}$$

$$A_y = A'_y \qquad A_z = A'_z$$

$$\varphi = \frac{(v/c)A'_x + \varphi'}{\sqrt{1-v^2/c^2}}$$

В вакууме

$$\mathbf{B} = \mathbf{H} = \operatorname{rot}\mathbf{A} \qquad \mathbf{E} = -\operatorname{grad}\varphi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}$$

Построим антисимметричный четырехтензор

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial r_\alpha} - \frac{\partial A_\alpha}{\partial r_\beta}$$

$$F_{11}=\frac{\partial A_1}{\partial r_1}-\frac{\partial A_1}{\partial r_1}=0 \qquad F_{22}=F_{33}=F_{44}=0$$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$F_{12}=\frac{\partial A_2}{\partial r_1}-\frac{\partial A_1}{\partial r_2}=\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}=(\operatorname{rot}\mathbf{A})_z=H_z$$

$$F_{13}=\frac{\partial A_3}{\partial r_1}-\frac{\partial A_1}{\partial r_3}=\frac{\partial A_z}{\partial x}-\frac{\partial A_x}{\partial z}=-(\operatorname{rot}\mathbf{A})_y=-H_y$$

$$F_{14}=\frac{\partial A_4}{\partial r_1}-\frac{\partial A_1}{\partial r_4}=\frac{i\partial\varphi}{\partial x}-\frac{\partial A_x}{ic\partial t}=i(\operatorname{grad}\varphi)_x+\frac{i}{c}\frac{\partial A_x}{\partial t}=-iE_x$$

$$F_{23}=\frac{\partial A_3}{\partial r_2}-\frac{\partial A_2}{\partial r_3}=\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}=(\operatorname{rot}\mathbf{A})_x=H_x$$

$$F_{24}=\frac{\partial A_4}{\partial r_2}-\frac{\partial A_2}{\partial r_4}=\frac{i\partial\varphi}{\partial y}-\frac{\partial A_y}{ic\partial t}=i(\operatorname{grad}\varphi)_y+\frac{i}{c}\frac{\partial A_y}{\partial t}=-iE_y$$

$$F_{34}=\frac{\partial A_4}{\partial r_3}-\frac{\partial A_3}{\partial r_4}=\frac{i\partial\varphi}{\partial z}-\frac{\partial A_z}{ic\partial t}=i(\operatorname{grad}\varphi)_z+\frac{i}{c}\frac{\partial A_z}{\partial t}=-iE_z$$

Четырехтензор электромагнитного поля

$$F_{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

Уравнения Максвелла

$$\frac{\partial F_{\alpha\beta}}{\partial r_\gamma} + \frac{\partial F_{\beta\gamma}}{\partial r_\alpha} + \frac{\partial F_{\gamma\alpha}}{\partial r_\beta} = 0 \qquad \frac{\partial F_{\alpha\beta}}{\partial r_\beta} = \frac{4\pi}{c} j_\alpha$$

α	β	γ
1	2	3
1	2	4
1	3	4
2	3	4

1. $$\frac{\partial F_{12}}{\partial r_3} + \frac{\partial F_{23}}{\partial r_1} + \frac{\partial F_{31}}{\partial r_2} = 0$$

$$\frac{\partial H_z}{\partial z} + \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} = 0$$

$$\boxed{\operatorname{div} \mathbf{H} = 0}$$

2. $$\frac{\partial F_{12}}{\partial r_4}+\frac{\partial F_{24}}{\partial r_1}+\frac{\partial F_{41}}{\partial r_2}=0$$

$$(\mathrm{rot}\,\mathbf{E})_z=-\frac{1}{c}\frac{\partial H_z}{\partial t}$$

$$\frac{\partial H_z}{ic\partial t}-\frac{i\partial E_y}{\partial x}+\frac{i\partial E_x}{\partial y}=0$$

$$F_{\alpha\beta}=\begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

3. $$\frac{\partial F_{13}}{\partial r_4}+\frac{\partial F_{34}}{\partial r_1}+\frac{\partial F_{41}}{\partial r_3}=0$$

$$-\frac{\partial H_y}{ic\partial t}-\frac{i\partial E_z}{\partial x}+\frac{i\partial E_x}{\partial z}=0$$

$$(\mathrm{rot}\,\mathbf{E})_y=-\frac{1}{c}\frac{\partial H_y}{\partial t}$$

4. $$\frac{\partial F_{23}}{\partial r_4}+\frac{\partial F_{34}}{\partial r_2}+\frac{\partial F_{42}}{\partial r_3}=0$$

$$(\mathrm{rot}\,\mathbf{E})_x=-\frac{1}{c}\frac{\partial H_x}{\partial t}$$

$$\frac{\partial H_x}{ic\partial t}-\frac{i\partial E_z}{\partial y}+\frac{i\partial E_y}{\partial z}=0$$

$$\boxed{\mathrm{rot}\,\mathbf{E}=-\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}}$$

$$\frac{\partial F_{\alpha\beta}}{\partial r_\beta} = \frac{4\pi}{c} j_\alpha$$

1. $$\frac{\partial F_{12}}{\partial r_2} + \frac{\partial F_{13}}{\partial r_3} + \frac{\partial F_{14}}{\partial r_4} = \frac{4\pi}{c} j_1 \qquad \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} - \frac{i\partial E_x}{ic\partial t} = \frac{4\pi}{c} j_x$$

$$(\mathrm{rot}\,\mathbf{H})_x = \frac{1}{c}\frac{\partial E_x}{\partial t} + \frac{4\pi}{c} j_x$$

2. $$\frac{\partial F_{21}}{\partial r_1} + \frac{\partial F_{23}}{\partial r_3} + \frac{\partial F_{24}}{\partial r_4} = \frac{4\pi}{c} j_2 \qquad -\frac{\partial H_z}{\partial x} + \frac{\partial H_x}{\partial z} - \frac{i\partial E_y}{ic\partial t} = \frac{4\pi}{c} j_y$$

$$(\mathrm{rot}\,\mathbf{H})_y = \frac{1}{c}\frac{\partial E_y}{\partial t} + \frac{4\pi}{c} j_y$$

3. $$\frac{\partial F_{31}}{\partial r_1} + \frac{\partial F_{32}}{\partial r_2} + \frac{\partial F_{34}}{\partial r_4} = \frac{4\pi}{c} j_3 \qquad \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} - \frac{i\partial E_z}{ic\partial t} = \frac{4\pi}{c} j_z$$

$$(\mathrm{rot}\,\mathbf{H})_z = \frac{1}{c}\frac{\partial E_z}{\partial t} + \frac{4\pi}{c} j_z$$

$$\boxed{\mathrm{rot}\,\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{j}}$$

4. $$\frac{\partial F_{41}}{\partial r_1}+\frac{\partial F_{42}}{\partial r_2}+\frac{\partial F_{43}}{\partial r_3}=\frac{4\pi}{c}j_4 \qquad \frac{i\partial E_x}{\partial x}+\frac{i\partial E_y}{\partial y}+\frac{i\partial E_z}{\partial z}=\frac{4\pi}{c}ic\rho$$

$$\operatorname{div}\mathbf{E}=4\pi\rho$$

Уравнения Максвелла

$$\boxed{\frac{\partial F_{\alpha\beta}}{\partial r_\gamma}+\frac{\partial F_{\beta\gamma}}{\partial r_\alpha}+\frac{\partial F_{\gamma\alpha}}{\partial r_\beta}=0 \qquad \frac{\partial F_{\alpha\beta}}{\partial r_\beta}=\frac{4\pi}{c}j_\alpha}$$

записаны в релятивистски инвариантной форме!!!

$$\frac{\partial F'_{\alpha\beta}}{\partial r'_\gamma}+\frac{\partial F'_{\beta\gamma}}{\partial r'_\alpha}+\frac{\partial F'_{\gamma\alpha}}{\partial r'_\beta}=0 \Rightarrow$$

1. $$\frac{\partial F_{12}}{\partial r_3}+\frac{\partial F_{23}}{\partial r_1}+\frac{\partial F_{31}}{\partial r_2}=0 \qquad i=4$$

2. $$\frac{\partial F_{12}}{\partial r_4}+\frac{\partial F_{24}}{\partial r_1}+\frac{\partial F_{41}}{\partial r_2}=0 \qquad i=3$$

$$e_{i\gamma\alpha\beta}\frac{\partial F_{\alpha\beta}}{\partial r_\gamma}=0 \Rightarrow$$

3. $$\frac{\partial F_{13}}{\partial r_4}+\frac{\partial F_{34}}{\partial r_1}+\frac{\partial F_{41}}{\partial r_3}=0 \qquad i=2$$

4. $$\frac{\partial F_{23}}{\partial r_4}+\frac{\partial F_{34}}{\partial r_2}+\frac{\partial F_{42}}{\partial r_3}=0 \qquad i=1$$

$$0 = e'_{i\mu\alpha\beta}\frac{\partial F'_{\alpha\beta}}{\partial r'_\mu} = \tilde\gamma_{ii'}\tilde\gamma_{\mu\mu'}\tilde\gamma_{\alpha\alpha'}\tilde\gamma_{\beta\beta'}\, e_{i'\mu'\alpha'\beta'}\,\gamma_{k\mu}\frac{\partial}{\partial r_k}\tilde\gamma_{\alpha m}\tilde\gamma_{\beta n}F_{mn} =$$

$\delta_{k\mu'}$

$$= \tilde\gamma_{ii'}\, e_{i'kmn}\frac{\partial}{\partial r_k}F_{mn}$$

$\delta_{\alpha' m}$ $\qquad$ $\delta_{\beta' n}$

$$\partial/\partial r'_\alpha = \partial/\partial r_\mu\ \gamma_{\mu\alpha}$$

$$F'_{\alpha\beta} = \tilde\gamma_{\alpha m}\tilde\gamma_{\beta n}F_{mn}$$

Равенство нулю достигается, если

$$e_{i'kmn}\frac{\partial}{\partial r_k}F_{mn} = 0$$

Второе уравнение $\dfrac{\partial F_{\alpha\beta}}{\partial r_\beta} = \dfrac{4\pi}{c}j_\alpha$

$$\frac{\partial F'_{\alpha\beta}}{\partial r'_\beta} = \frac{\partial}{\partial r_\mu}\gamma_{\mu\beta}F'_{\alpha\beta} = \frac{\partial}{\partial r_\mu}\gamma_{\mu\beta}\tilde\gamma_{\alpha\varsigma}\tilde\gamma_{\beta\xi}F_{\varsigma\xi} = \frac{4\pi}{c}\tilde\gamma_{\alpha\varsigma}j_\varsigma$$

$$\gamma_{k\alpha}\frac{\partial}{\partial r_\mu}\gamma_{\mu\beta}\tilde\gamma_{\alpha\varsigma}\tilde\gamma_{\beta\xi}F_{\varsigma\xi} = \gamma_{k\alpha}\frac{4\pi}{c}\tilde\gamma_{\alpha\varsigma}j_\varsigma \qquad \delta_{k\varsigma}\frac{\partial}{\partial r_\mu}\delta_{\mu\xi}F_{\varsigma\xi} = \delta_{k\varsigma}\frac{4\pi}{c}j_\varsigma$$

$$\frac{\partial}{\partial r_\mu}F_{k\mu} = \frac{4\pi}{c}j_k$$

Преобразование электромагнитного поля

$$F_{\alpha\beta} = \gamma_{\alpha\varsigma}\gamma_{\beta\xi}F'_{\varsigma\xi}$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \dfrac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{-iv/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \dfrac{iv/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \dfrac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}$$

$$F_{12} = \gamma_{1\varsigma}\gamma_{2\xi}F'_{\varsigma\xi} = \gamma_{11}\gamma_{2\xi}F'_{1\xi} + \gamma_{14}\gamma_{2\xi}F'_{4\xi} = \gamma_{11}\gamma_{22}F'_{12} + \gamma_{14}\gamma_{22}F'_{42} =$$

$$= \frac{F'_{12} - i(v/c)F'_{42}}{\sqrt{1-v^2/c^2}}$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

$$\boxed{H_z = \frac{H'_z + (v/c)E'_y}{\sqrt{1-v^2/c^2}}}$$

$$F_{13} = \gamma_{1\varsigma}\gamma_{3\xi}F'_{\varsigma\xi} = \gamma_{11}\gamma_{3\xi}F'_{1\xi} + \gamma_{14}\gamma_{3\xi}F'_{4\xi} = \gamma_{11}\gamma_{33}F'_{13} + \gamma_{14}\gamma_{33}F'_{43} =$$

$$= \frac{F'_{13} - i(v/c)F'_{43}}{\sqrt{1-v^2/c^2}} \qquad -H_y = \frac{-H'_y - i(v/c)iE'_z}{\sqrt{1-v^2/c^2}} \qquad \boxed{H_y = \frac{H'_y - (v/c)E'_z}{\sqrt{1-v^2/c^2}}}$$

$$F_{14} = \gamma_{1\varsigma}\gamma_{4\xi}F'_{\varsigma\xi} = \gamma_{11}\gamma_{4\xi}F'_{1\xi} + \gamma_{14}\gamma_{4\xi}F'_{4\xi} =$$

$$= \cancel{\gamma_{11}\gamma_{41}F'_{11}} + \gamma_{11}\gamma_{44}F'_{14} + \gamma_{14}\gamma_{41}F'_{41} + \cancel{\gamma_{14}\gamma_{44}F'_{44}} = \frac{F'_{14} + (v/c)^2 F'_{41}}{1-v^2/c^2}$$

$$-iE_x = \frac{-iE'_x + (v/c)^2 iE'_x}{1-v^2/c^2} \qquad \boxed{E_x = E'_x}$$

$$F_{23} = \gamma_{2\varsigma}\gamma_{3\xi}F'_{\varsigma\xi} = \gamma_{22}\gamma_{3\xi}F'_{2\xi} = \gamma_{22}\gamma_{33}F'_{23} \qquad \boxed{H_x = H'_x}$$

$$F_{24} = \gamma_{2\varsigma}\gamma_{4\xi}F'_{\varsigma\xi} = \gamma_{22}\gamma_{4\xi}F'_{2\xi} = \gamma_{11}\gamma_{41}F'_{21} + \gamma_{22}\gamma_{44}F'_{24}$$

$$-iE_y = \frac{-i(v/c)H'_z - iE'_y}{\sqrt{1-v^2/c^2}} \qquad \boxed{E_y = \frac{(v/c)H'_z + E'_y}{\sqrt{1-v^2/c^2}}}$$

$$F_{34} = \gamma_{3\varsigma}\gamma_{4\xi}F'_{\varsigma\xi} = \gamma_{33}\gamma_{4\xi}F'_{3\xi} = \gamma_{33}\gamma_{41}F'_{31} + \gamma_{33}\gamma_{44}F'_{34} =$$

$$-iE_z = \frac{i(v/c)H'_y - iE'_z}{\sqrt{1-v^2/c^2}}$$

$$\boxed{E_z = \frac{-(v/c)H'_y + E'_z}{\sqrt{1-v^2/c^2}}}$$

$$E_x = E'_x$$

$$E_y = \frac{E'_y + (v/c)H'_z}{\sqrt{1-v^2/c^2}}$$

$$E_z = \frac{E'_z - (v/c)H'_y}{\sqrt{1-v^2/c^2}}$$

$$H_x = H'_x$$

$$H_y = \frac{H'_y - (v/c)E'_z}{\sqrt{1-v^2/c^2}}$$

$$H_z = \frac{H'_z + (v/c)E'_y}{\sqrt{1-v^2/c^2}}$$

$v << c$

$$E_x = E'_x$$

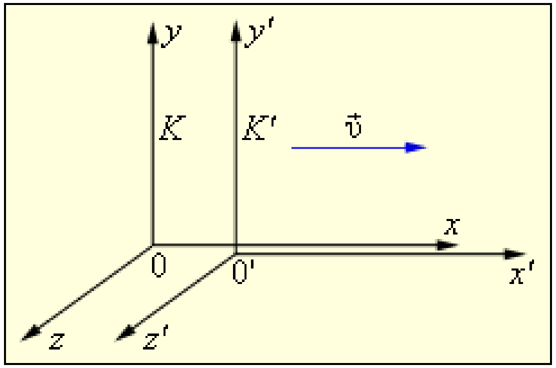
$$E_y = E'_y + (v/c)H'_z$$

$$E_z = E'_z - (v/c)H'_y$$

$$H_x = H'_x$$

$$H_y = H'_y - (v/c)E'_z$$

$$H_z = H'_z + (v/c)E'_y$$



В случае малых скоростей

$$\mathbf{E} = \mathbf{E}' - \frac{1}{c}[\mathbf{vH}'] \quad \mathbf{H} = \mathbf{H}' + \frac{1}{c}[\mathbf{vE}'] \qquad \mathbf{E}' = \mathbf{E} + \frac{1}{c}[\mathbf{vH}] \quad \mathbf{H}' = \mathbf{H} - \frac{1}{c}[\mathbf{vE}]$$

Инварианты электромагнитного поля

$$I_1 = \mathrm{Sp}\,\hat{F} \qquad I_2 = F_{\alpha\beta}F_{\alpha\beta} \qquad I_3 = F_{\alpha\beta}F_{\beta\delta}F_{\delta\alpha} \qquad I_4 = \det\hat{F}$$

$$I_1 = 0$$

$$I_2 = H_z^2 + H_y^2 - E_x^2 +$$

$$+ H_x^2 - E_y^2 - E_z^2 = \mathbf{H}^2 - \mathbf{E}^2$$

$$F_{\alpha\beta} = \begin{pmatrix} 0 & H_z & -H_y & -iE_x \\ -H_z & 0 & H_x & -iE_y \\ H_y & -H_x & 0 & -iE_z \\ iE_x & iE_y & iE_z & 0 \end{pmatrix}$$

$$I_3 = F_{1\beta}F_{\beta\delta}F_{\delta 1} + F_{2\beta}F_{\beta\delta}F_{\delta 2} + F_{3\beta}F_{\beta\delta}F_{\delta 3} + F_{4\beta}F_{\beta\delta}F_{\delta 4}$$

$$F_{1\beta}F_{\beta\delta}F_{\delta 1} = \underline{F_{12}F_{2\delta}F_{\delta 1}} + \underline{F_{13}F_{3\delta}F_{\delta 1}} + F_{14}F_{4\delta}F_{\delta 1} = \underline{F_{12}F_{23}F_{31} + F_{12}F_{24}F_{41}}$$

$$+ \underline{F_{13}F_{32}F_{21} + F_{13}F_{34}F_{41}} + F_{14}F_{42}F_{21} + F_{14}F_{43}F_{31} =$$

$$F_{1\beta}F_{\beta\delta}F_{\delta 1} = \cancel{F_{12}F_{23}F_{31}} + \cancel{F_{12}F_{24}F_{41}} + \cancel{F_{13}F_{32}F_{21}} + F_{13}F_{34}F_{41} + \cancel{F_{14}F_{42}F_{21}} +$$

$$+ F_{14}F_{43}F_{31} = 0$$

$$F_{2\beta}F_{\beta\delta}F_{\delta 2} = F_{3\beta}F_{\beta\delta}F_{\delta 3} = F_{4\beta}F_{\beta\delta}F_{\delta 4} = 0 \qquad\qquad I_3 = 0$$

$$I_4 = \det \hat{F} = -H_z \begin{pmatrix} -H_z & H_x & -iE_y \\ H_y & 0 & -iE_z \\ iE_x & iE_z & 0 \end{pmatrix} - H_y \begin{pmatrix} -H_z & 0 & -iE_y \\ H_y & -H_x & -iE_z \\ iE_x & iE_y & 0 \end{pmatrix}$$

$$+iE_x \begin{pmatrix} -H_z & 0 & H_x \\ H_y & -H_x & 0 \\ iE_x & iE_y & iE_z \end{pmatrix} = -H_z[H_z E_z^2 + H_x E_x E_z + H_y E_z E_y] +$$

$$+H_y[H_z E_y E_z + H_y E_y^2 + H_x E_y E_x] + iE_x[iE_z H_z H_x + H_x H_y iE_y +$$

$$+H_x H_x iE_x] = -H_z^2 E_z^2 - \underline{H_z H_x E_x E_z} - \underline{H_z H_y E_z E_y} +$$

$$\underline{-H_y H_z E_y E_z} - H_y^2 E_y^2 - H_y H_x E_y E_x - \underline{E_x E_z H_z H_x} - E_x H_x H_y E_y +$$

$$-E_x^2 H_x^2 = -(E_x H_x + E_y H_y + E_z H_z)^2 = -(\mathbf{EH})^2$$

Абсолютные характеристики электромагнитного поля

$$I_2 = \mathbf{H}^2 - \mathbf{E}^2 \qquad\qquad I_4 = -(\mathbf{EH})^2$$

Например, если электрическое и магнитное поле в некоторой инерциальной системе отсчета перпендикулярны друг другу и $\mathbf{H}^2 > \mathbf{E}^2$, то можно найти систему отсчета в которой электрическое поле отсутствует.

Релятивистская природа силы Лоренца

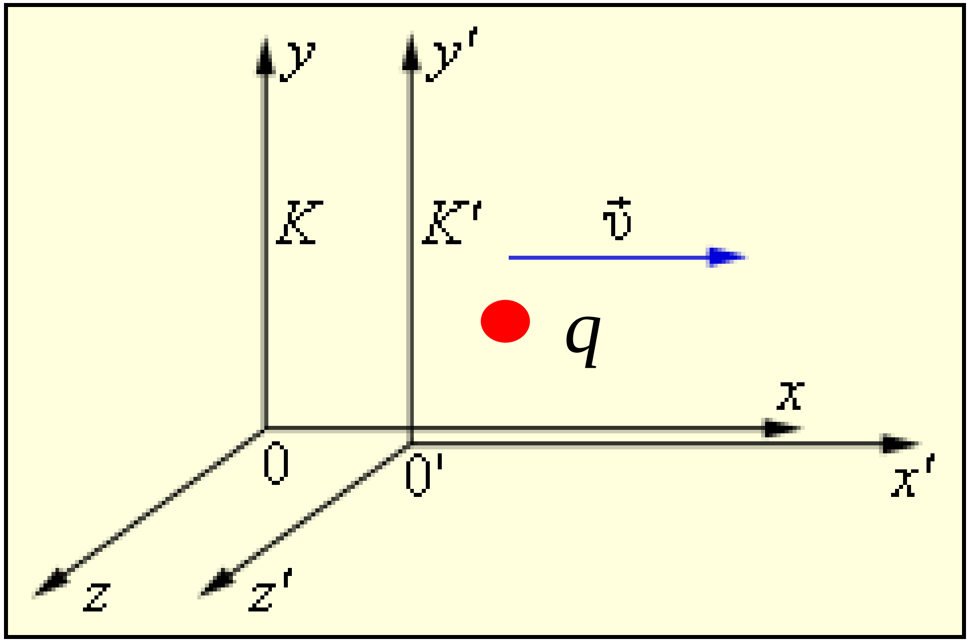
Четырехвектор силы

$$f_\alpha = \left\{ \frac{f_x}{\sqrt{1-v^2/c^2}}, \frac{f_y}{\sqrt{1-v^2/c^2}}, \frac{f_z}{\sqrt{1-v^2/c^2}}, \frac{i(\mathbf{fv})}{c\sqrt{1-v^2/c^2}} \right\}$$

$$f_\alpha = \gamma_{\alpha\beta} f'_\beta$$

Пусть в системе К’ заряд q покоится

$$f'_\alpha = \left\{ qE'_x, qE'_y, qE'_z, 0 \right\}$$



$$f'_\alpha = \left\{ qE'_x, qE'_y, qE'_z, 0 \right\}$$

$$\gamma_{\alpha\beta} = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{-iv/c}{\sqrt{1-v^2/c^2}} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{iv/c}{\sqrt{1-v^2/c^2}} & 0 & 0 & \frac{1}{\sqrt{1-v^2/c^2}} \end{pmatrix}$$

$$f'_4 = 0 \qquad (E'_x = E_x)$$

$$f_1 = \frac{f_x}{\sqrt{1-v^2/c^2}} = \gamma_{11} f'_1 + \gamma_{14} f'_4 = \frac{1}{\sqrt{1-v^2/c^2}} qE'_x = \frac{1}{\sqrt{1-v^2/c^2}} qE_x \Rightarrow f_x = qE_x$$

$$f_2 = \frac{f_y}{\sqrt{1-v^2/c^2}} = \gamma_{22} f'_2 = qE'_y = q\frac{E_y - (v/c)H_z}{\sqrt{1-v^2/c^2}} \Rightarrow f_y = q[E_y - (v/c)H_z]$$

$$f_3 = \frac{f_z}{\sqrt{1-v^2/c^2}} = \gamma_{33} f'_3 = qE'_z = q\frac{E_z + (v/c)H_y}{\sqrt{1-v^2/c^2}} \Rightarrow f_z = q[E_z + (v/c)H_y]$$

$$\mathbf{f} = q\mathbf{E} + \frac{q}{c}[\mathbf{vH}]$$

$$\Delta E - \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2} = 0 \qquad E = u(x,y,z)\exp(-i\omega t)$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

$$u(x,y,z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(k_x,k_y,z)\exp(ik_x x + ik_y y)dk_x dk_y$$

$$\frac{\partial^2 F}{\partial z^2} + (k^2 - k_x^2 - k_y^2)F = 0$$

$$F(k_x,k_y,z) = F_0(k_x,k_y)\exp\left(iz\sqrt{k^2 - k_x^2 - k_y^2}\right) \approx$$

$$\approx F_0(k_x,k_y)\exp(ikz)\exp\left[-\frac{i}{2k}(k_x^2 + k_y^2)z\right]$$

$$F_0(k_x,k_y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(\varsigma,\eta,0)\exp(-ik_x\varsigma - ik_y\eta)d\varsigma d\eta$$

$$u(x,y,z)=\exp(ikz)\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\varsigma,\eta,0)\exp(-ik_x\varsigma-ik_y\eta)d\varsigma d\eta\exp(ik_x x+ik_y y)\times$$

$$\times\exp\left[-\frac{i}{2k}(k_x^2+k_y^2)z\right]dk_x dk_y=A(x,y,z)\exp(ikz)$$

$$\int_{-\infty}^{\infty}\exp\left[ik_x(x-\varsigma)-\frac{iz}{2k}k_x^2\right]dk_x=\sqrt{\frac{\pi k}{z}}(1-i)\exp\left[\frac{ik}{2z}(x-\varsigma)^2\right]$$

$$\int_{-\infty}^{\infty}\exp\left[ik_x(x-\varsigma)\right]dk_x=2\pi\delta(x-\varsigma)$$

$$u(x,y,z)=\exp(ikz)\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\varsigma,\eta,0)\sqrt{\frac{\pi k}{z}}(1-i)\exp\left[\frac{ik}{2z}(x-\varsigma)^2\right]\sqrt{\frac{\pi k}{z}}(1-i)\exp\left[\frac{ik}{2z}(y-\eta)^2\right]d\varsigma d\eta=$$

$$=\left\{-\frac{ik}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\varsigma,\eta,0)\exp\left[\frac{ik}{2z}[(x-\varsigma)^2+(y-\eta)^2]\right]d\varsigma d\eta\right\}\exp(ikz)=A(x,y,z)\exp(ikz)$$

$$A(x,y,z)=-\frac{ik}{2\pi z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\varsigma,\eta,0)\exp\left[\frac{ik}{2z}[(x-\varsigma)^2+(y-\eta)^2]\right]d\varsigma d\eta$$

$$A(x,y,0)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\varsigma,\eta,0)\exp(-ik_x\varsigma-ik_y\eta)\,d\varsigma\, d\eta\exp(ik_x x+ik_y y)\,dk_x\, dk_y=$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u(\varsigma,\eta,0)\,d\varsigma\, d\eta\,\delta(x-\varsigma)\delta(y-\eta)=u(x,y,0)$$

$$2ik\frac{\partial A}{\partial z}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}+k^2u=0 \qquad u(x,y,z)=A(\sqrt{\mu}x,\sqrt{\mu}y,\mu z)\exp(ikz)$$

$$\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}+\cancel{\frac{\partial^2 A}{\partial z^2}}+2ik\frac{\partial A}{\partial z}-k^2A+k^2A\right)=0$$

$$A(x,y,0) = A_0 \exp\left(-\frac{x^2 + y^2}{a^2} \right)$$

$$A(x,y,z) = -\frac{ikA_0}{2\pi z} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{\varsigma^2 + \eta^2}{a^2} + \frac{ik}{2z}[(x-\varsigma)^2 + (y-\eta)^2] \right] d\varsigma d\eta =$$

$$= \frac{A_0}{1 + \frac{2iz}{ka^2}} \exp\left[-\frac{x^2 + y^2}{a^2\left(1 + \frac{2iz}{ka^2}\right)} \right] =$$

$$= \frac{A_0}{\sqrt{1 + z^2 / L^2}} \exp\left[-\frac{x^2 + y^2}{a^2\left(1 + z^2 / L^2\right)} + \frac{ik(x^2 + y^2)z}{a^2\left(1 + z^2 / L^2\right)L} - i \operatorname{arctg}\left(\frac{z}{L} \right) \right]$$

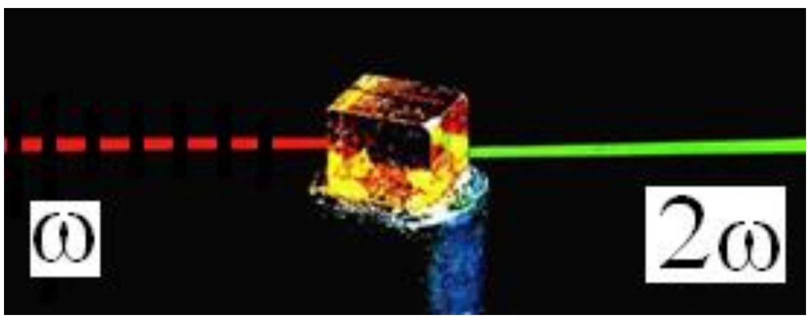
$$a(z) = a(0)\left(1 + z^2 / L^2\right)^{1/2} \qquad L = ka^2$$

Материальное уравнение

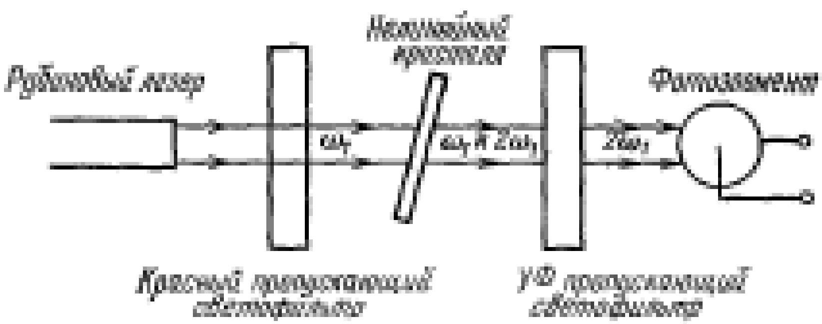
$$P_i = \chi_{ij}^{(1)} E_j + \chi_{ijl}^{(2)} E_j E_l + \chi_{ijlm}^{(3)} E_j E_l E_m + \ldots$$

Генерация второй гармоники и суммарной частота

Генерация третьей гармоники, CARS, самофокусировка, НОА



Генерация второй гармоники



P. Franken et al., 1961, Phys. Rev. Letts.

Самофокусировка лазерного пучка

«Воздействие луча на среду может быть настолько сильным, что создается перепад свойств среды в луче и вне луча, что вызовет волноводное распространение луча и устранит геометрическую и дифракционную расходимость. Это интересное явление можно назвать самофокусировкой электромагнитного луча»

ЖЭТФ, 42, 1567, 1962

OBSERVATION OF SELF-FOCUSING OF LIGHT IN LIQUIDS

N. F. Pilipetskii and A. R. Rustamov
Moscow State University
Submitted 31 May 1965

В 1965 году выпускники факультета Николай Федорович Пилипецкий и Саидазим Рустамович Рустамов впервые наблюдали самофокусировку экспериментально.

In 1962 G. A. Askar'yan considered one of the important problems involved in the effect of a beam of intense radiation on a medium. He has shown that intense radiation can lead to a differential between the properties of the medium inside and outside the beam. The latter creates conditions suitable for waveguide propagation of the beam, thereby eliminating the geometrical and diffractional divergences. This interesting phenomenon was called by him self-focusing of an electromagnetic beam.

Самофокусировка линейно поляризованного лазерного пучка

$$\varepsilon = \varepsilon_0 + \varepsilon_{nl} \,|\, A|^2$$

$$2ik \frac{\partial A}{\partial z} = \Delta_\perp A + \frac{k^2 \varepsilon_{nl} \,|\, A|^2}{\varepsilon_0} A$$

$$-2ik \frac{\partial A^*}{\partial z} = \Delta_\perp A^* + \frac{k^2 \varepsilon_{nl} \,|\, A|^2}{\varepsilon_0} A^*$$

$$2ik \frac{\partial A^*}{\partial z} \frac{\partial A}{\partial z} = \frac{\partial A^*}{\partial z} \Delta_\perp A + \frac{k^2 \varepsilon_{nl} \,|\, A|^2}{\varepsilon_0} A \frac{\partial A^*}{\partial z}$$

$$-2ik \frac{\partial A}{\partial z} \frac{\partial A^*}{\partial z} = \frac{\partial A}{\partial z} \Delta_\perp A^* + \frac{k^2 \varepsilon_{nl} \,|\, A|^2}{\varepsilon_0} \frac{\partial A}{\partial z} A^*$$

$$0 = \frac{\partial A^*}{\partial z} \Delta_\perp A + \frac{\partial A}{\partial z} \Delta_\perp A^* + \frac{k^2 \varepsilon_{nl} \,|\, A|^2}{\varepsilon_0} \left(A \frac{\partial A^*}{\partial z} + A^* \frac{\partial A}{\partial z} \right)$$

$$0=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial A^*}{\partial z}\left(\frac{\partial^2 A}{\partial x^2}+\frac{\partial^2 A}{\partial y^2}\right)dxdy+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial A}{\partial z}\left(\frac{\partial^2 A^*}{\partial x^2}+\frac{\partial^2 A^*}{\partial y^2}\right)dxdy$$

$$+\frac{k^2\varepsilon_{nl}}{2\varepsilon_0}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A|^4\,dxdy$$

Граничные условия

$$0=\cancel{\int_{-\infty}^{\infty}\frac{\partial A^*}{\partial z}\frac{\partial A}{\partial x}\bigg|_{-\infty}^{\infty}}dy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A^*}{\partial z\partial x}\frac{\partial A}{\partial x}dxdy+\cancel{\int_{-\infty}^{\infty}\frac{\partial A^*}{\partial z}\frac{\partial A}{\partial y}\bigg|_{-\infty}^{\infty}}dx-$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A^*}{\partial z\partial y}\frac{\partial A}{\partial y}dxdy+\cancel{\int_{-\infty}^{\infty}\frac{\partial A}{\partial z}\frac{\partial A^*}{\partial x}\bigg|_{-\infty}^{\infty}}dy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A}{\partial z\partial x}\frac{\partial A^*}{\partial x}dxdy+$$

$$+\cancel{\int_{-\infty}^{\infty}\frac{\partial A}{\partial z}\frac{\partial A^*}{\partial y}\bigg|_{-\infty}^{\infty}}dx-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A}{\partial z\partial y}\frac{\partial A^*}{\partial y}dxdy+\frac{k^2\varepsilon_{nl}}{2\varepsilon_0}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A|^4\,dxdy$$

$$=0$$

$$0=-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A^*}{\partial z\partial x}\frac{\partial A}{\partial x}dxdy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A^*}{\partial z\partial y}\frac{\partial A}{\partial y}dxdy-$$

$$-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A}{\partial z\partial x}\frac{\partial A^*}{\partial x}dxdy-\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\partial^2 A}{\partial z\partial y}\frac{\partial A^*}{\partial y}dxdy+\frac{k^2\varepsilon_{nl}}{2\varepsilon_0}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A|^4\,dxdy$$

$$0=-\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial A}{\partial x}\right|^2 dxdy-\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left|\frac{\partial A}{\partial y}\right|^2 dxdy+\frac{k^2\varepsilon_{nl}}{2\varepsilon_0}\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|A|^4\,dxdy$$

$$0=\frac{\partial}{\partial z}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left|\frac{\partial A}{\partial x}\right|^2+\left|\frac{\partial A}{\partial y}\right|^2-\frac{k^2\varepsilon_{nl}\,|A|^4}{2\varepsilon_0}\right]dxdy\right\}$$

$$A(r,z)=\frac{E_0}{f(z)}\exp\left\{-\frac{r^2}{r_0^2 f^2(z)}-\frac{ikr^2}{2f}\frac{\partial f}{\partial z}+i\varphi(z)\right\}$$

$$\frac{\partial A}{\partial x}=\frac{E_0}{f(z)}\left\{-\frac{2x}{r_0^2 f^2(z)}-\frac{i2xk}{2f}\frac{\partial f}{\partial z}\right\}\exp\left\{-\frac{r^2}{r_0^2 f^2(z)}-\frac{ikr^2}{2f}\frac{\partial f}{\partial z}+i\varphi(z)\right\}$$

$$\left|\frac{\partial A}{\partial x}\right|^2=\frac{E_0^2}{f^2(z)}\left\{4x^2\left(\frac{1}{r_0^4 f^4(z)}+\frac{k^2}{4f^2}\left(\frac{\partial f}{\partial z}\right)^2\right)\right\}\exp\left\{-\frac{2r^2}{r_0^2 f^2(z)}\right\}$$

$$I=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left|\frac{\partial A}{\partial x}\right|^2+\left|\frac{\partial A}{\partial y}\right|^2\right\}dxdy=$$

$$=\int_0^{\infty}\int_0^{2\pi}\frac{E_0^2}{f^2(z)}\left\{4r^2\left(\frac{1}{r_0^4 f^4(z)}+\frac{k^2}{4f^2}\left(\frac{\partial f}{\partial z}\right)^2\right)\right\}\exp\left\{-\frac{2r^2}{r_0^2 f^2(z)}\right\}rdrd\phi=$$

$$=\frac{4\pi E_0^2}{f^2(z)}\left(\frac{1}{r_0^4 f^4(z)}+\frac{k^2}{4f^2}\left(\frac{\partial f}{\partial z}\right)^2\right)\int_0^{\infty}r^2\exp\left\{-\frac{2r^2}{r_0^2 f^2(z)}\right\}dr^2$$

$$\int_0^\infty r^2 \exp\left\{-\frac{2r^2}{r_0^2 f^2(z)}\right\} dr^2 = \int_0^\infty u \exp\left\{-\frac{2u}{r_0^2 f^2(z)}\right\} du =$$

$$= -u\frac{r_0^4 f^4(z)}{4}\exp\left(-\frac{2u}{r_0^2 f^2(z)}\right)\Bigg|_0^\infty - \frac{r_0^4 f^4(z)}{4}\exp\left(-\frac{2u}{r_0^2 f^2(z)}\right)\Bigg|_0^\infty = \frac{r_0^4 f^4(z)}{4}$$

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\left|\frac{\partial A}{\partial x}\right|^2 + \left|\frac{\partial A}{\partial y}\right|^2\right\} dxdy = \frac{4\pi E_0^2}{f^2(z)}\left(\frac{1}{r_0^4 f^4(z)} + \frac{k^2}{4f^2}\left(\frac{\partial f}{\partial z}\right)^2\right)\int_0^\infty r^2 \exp\left\{-\frac{2r^2}{r_0^2 f^2(z)}\right\} dr^2 =$$

$$= \frac{4\pi E_0^2}{f^2(z)}\left(\frac{1}{r_0^4 f^4(z)} + \frac{k^2}{4f^2}\left(\frac{\partial f}{\partial z}\right)^2\right)\frac{r_0^4 f^4(z)}{4} = \frac{\pi E_0^2}{f^2(z)}\left(1 + \frac{k^2 r_0^4 f^2(z)}{4}\left(\frac{\partial f}{\partial z}\right)^2\right)$$

$$-\frac{k^2\varepsilon_{nl}}{2\varepsilon_0}\frac{E_0^4}{f^4(z)}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left\{-\frac{4r^2}{r_0^2 f^2(z)}\right\} dxdy = -\frac{k^2\varepsilon_{nl}\pi}{2\varepsilon_0}\frac{E_0^4}{f^4(z)}\times$$

$$\times\int_0^\infty \exp\left\{-\frac{4r^2}{r_0^2 f^2(z)}\right\} dr^2 = -\frac{k^2\varepsilon_{nl}\pi}{2\varepsilon_0}\frac{E_0^4 r_0^2 f^2(z)}{4 f^4(z)} = -\frac{k^2\varepsilon_{nl}\pi}{8\varepsilon_0}\frac{E_0^4 r_0^2}{f^2(z)}$$

$$0=\frac{\partial}{\partial z}\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\left|\frac{\partial A}{\partial x}\right|^2+\left|\frac{\partial A}{\partial y}\right|^2-\frac{k^2\varepsilon_{nl}\,|A|^4}{2\varepsilon_0}\right]dxdy\right\}$$

$$0=\frac{\partial}{\partial z}\left\{\frac{\pi E_0^2}{f^2(z)}\left(1+\frac{k^2r_0^4f^2(z)}{4}\left(\frac{\partial f}{\partial z}\right)^2\right)-\frac{k^2\varepsilon_{nl}\pi}{8\varepsilon_0}\frac{E_0^4r_0^2}{f^2(z)}\right\}$$

$$0=\frac{\partial}{\partial z}\left\{\frac{1}{f^2(z)}+\frac{k^2r_0^4}{4}\left(\frac{\partial f}{\partial z}\right)^2-\frac{k^2\varepsilon_{nl}}{8\varepsilon_0}\frac{E_0^2r_0^2}{f^2(z)}\right\}$$

$$0=-\frac{1}{f^3(z)}\frac{\partial f}{\partial z}+\frac{k^2r_0^4}{4}\frac{\partial f}{\partial z}\frac{\partial^2 f}{\partial z^2}+\frac{k^2\varepsilon_{nl}}{8\varepsilon_0}\frac{E_0^2r_0^2}{f^3(z)}\frac{\partial f}{\partial z}$$

$$0=-\frac{1}{f^3(z)}+\frac{k^2r_0^4}{4}\frac{\partial^2 f}{\partial z^2}+\frac{k^2\varepsilon_{nl}}{8\varepsilon_0}\frac{E_0^2r_0^2}{f^3(z)}$$

$$L_d=kr_0^2/2$$

$$\frac{\partial^2 f}{\partial z^2}=\frac{1}{f^3(z)}\left(\frac{1}{L_d^2}-\frac{1}{L_{nl}^2}\right)$$

$$L_{nl}=r_0\sqrt{2\varepsilon_0/\varepsilon_{nl}E_0^2}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{f^3(z)}\left(\frac{1}{L_d^2} - \frac{1}{L_{nl}^2}\right) \qquad f(0) = 1 \qquad (df/dz)_{z=0} = 0 \qquad z_1 = z / L_d$$

$$\frac{\partial^2 f}{\partial z_1^2} = \frac{1}{f^3(z)}\left(1 - \frac{L_d^2}{L_{nl}^2}\right) = \frac{1}{f^3(z_1)}\left(1 - \frac{P_0}{P_{cr}}\right) \qquad P_0 = \frac{cn_0 E_0^2 r_0^2}{16}, \quad P_{cr} = \left(\frac{cn\varepsilon_0}{2k^2\varepsilon_{nl}}\right)$$

$$\left(\frac{df}{dz_1}\right)^2 = \left(1 - \frac{P_0}{P_{cr}}\right)\left(1 - \frac{1}{f^2}\right) \Rightarrow \left(\frac{df}{dz_1}\right)^2 + f\frac{d^2 f}{dz_1^2} = \left(1 - \frac{P_0}{P_{cr}}\right)$$

$$\left(\frac{df}{dz_1}\right)^2 + f\frac{d^2 f}{dz_1^2} = \frac{1}{2}\frac{d^2 f^2}{dz_1^2} \qquad \frac{df^2}{dz_1} = 2\left(1 - \frac{P_0}{P_{cr}}\right) z_1 \qquad f^2 = \left(1 - \frac{P_0}{P_{cr}}\right) z_1^2 + 1$$